MATH 304 ASSIGNMENT 11

All problems are from Leon’s *Linear Algebra*, eighth edition, unless otherwise specified. Turning in extra problems will *not* result in any extra credit.

0. NOT TO BE TURNED IN

The following problems are meant to test your understanding but are not to be turned in.
- Section 5.3: Problems 1, 2, 3, 4, 9.
- Section 5.4: Problems 1, 2, 3, 4, 17.

1. TO BE TURNED IN

Please complete these problems on a separate sheet of paper and hand them in.

1. (Section 5.3, problems 1c and 2c) Consider the following system:

\[
\begin{align*}
    x_1 + x_2 + x_3 &= 4 \\
    -x_1 + x_2 + x_3 &= 0 \\
    -x_2 + x_3 &= 1 \\
    x_1 + x_3 &= 2
\end{align*}
\]

(a) Find the least squares solution of the system.
(b) For your solution above, determine the projection \( p = A\hat{x} \).
(c) Calculate the residual \( r(\hat{x}) \).
(d) Verify that \( r(\hat{x}) \in N(A^T) \).

2. (Section 5.3, problems 5 and 6) Consider the following data:

\[
\begin{array}{c|cccc}
    x & -1 & 0 & 1 & 2 \\
    y & 0 & 1 & 3 & 9 \\
\end{array}
\]

(a) Find the best least squares fit by a linear function to the data above.
(b) Plot your linear function from part (a) along with the data on a coordinate system.
(c) Find the best least squares fit to the data above by a *quadratic* polynomial.
(d) Plot the points for your function and sketch the graph.

3. (Section 5.3, problem 10) Let \( A \) be an \( 8 \times 5 \) matrix of rank 3 and let \( b \) be a nonzero vector in \( N(A^T) \).
(a) Show that the system \( A\mathbf{x} = \mathbf{b} \) must be inconsistent.
(b) How many least squares solutions will the system \( A\mathbf{x} = \mathbf{b} \) have? Explain.

4. (Section 5.4, problem 7) Equip \( C[0,1] \) (the vector space of continuous functions on the interval \([0,1]\)) with the inner product

\[
    \langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.
\]

Compute
(a) \( \langle e^x, e^{-x} \rangle \),
(b) \( \langle x, \sin(\pi x) \rangle \),

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5. (Section 5.4, problem 11) For $i = 1, \ldots, 5$, let $x_i = (i - 3)/2$. We equip $P_5$ (the vector space of polynomials of degree at most 4) with an inner product given by

$$\langle p, q \rangle = \sum_{i=1}^{5} p(x_i)q(x_i),$$

and norm given by

$$\|p\| = \sqrt{\langle p, p \rangle} = \left( \sum_{i=1}^{5} p(x_i)^2 \right)^{1/2}.$$

Compute
(a) $\|x\|$, 
(b) $\|x^2\|$, 
(c) the distance between $x$ and $x^2$.

6. (Section 5.4, problem 18) Show that if $u$ and $v$ are vectors in an inner product space that satisfy the Pythagorean law

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

then $u$ and $v$ must be orthogonal.

7. (Section 5.4, problem 26) Prove that, for any $u$ and $v$ in an inner product space $V$,

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

2. Extra practice

These problems are suggested in case you would like extra practice with the concepts from class. They are not to be turned in.

- Section 5.3: Problems 11, 12, 13, 14.
- Section 5.4: Problems 20, 28, 30, 32, 33.