MATH 304 ASSIGNMENT 12

All problems are from Leon’s Linear Algebra, eighth edition, unless otherwise specified. Turning in extra problems will not result in any extra credit.

0. NOT TO BE TURNED IN

The following problems are meant to test your understanding but are not to be turned in.
- Section 5.5: Problems 1, 2, 5, 11, 15.
- Section 5.6: Problems 1, 2.

1. TO BE TURNED IN

Please complete these problems on a separate sheet of paper and hand them in.

1. (Section 5.5, problem 4) Let $\theta$ be a fixed real number and let
   $$x_1 = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad x_2 = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$
   (a) Show that $\{x_1, x_2\}$ is an orthonormal basis for $\mathbb{R}^2$.
   (b) Given an vector $y$ in $\mathbb{R}^2$, write it as a linear combination $c_1 x_1 + c_2 x_2$.
   (c) Verify that $c_1^2 + c_2^2 = \|y\|^2 = y_1^2 + y_2^2$.

2. (Section 5.5, problem 7) Let $\{u_1, u_2, u_3\}$ be an orthonormal basis for an inner product space $V$. If $x = c_1 u_1 + c_2 u_2 + c_3 u_3$ is a vector with the properties $\|x\| = 5$, $\langle x, u_1 \rangle = 4$, and $x \perp u_2$, then what are the possible values of $c_1$, $c_2$, and $c_3$?

3. (Section 5.5, problem 14) Let $u$ be a unit vector in $\mathbb{R}^n$ and let $H = I - 2uu^\top$. Show that $H$ is both orthogonal and symmetric and hence its own inverse.

4. (Section 5.5, problems 21–22) Let
   $$A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$
   (a) Show that the column vectors of $A$ form an orthonormal set in $\mathbb{R}^4$.
   (b) Solve the least squares problem $Ax = b$ for each of the following choices of $b$:
      (i) $b = (4, 0, 0, 0)^\top$
      (ii) $b = (1, 2, 3, 4)^\top$
      (iii) $b = (1, 1, 2, 2)^\top$
   (c) Find the projection matrix $P$ that projects vectors in $\mathbb{R}^4$ onto $C(A)$ (the column space of $A$).
   (d) For each of your solutions $\hat{x}$ to part (b), compute $Ax$ and compare it with $Pb$.

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5. (Section 5.6, problem 3) Given the basis
\[
\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}
\]
for \( \mathbb{R}^3 \), use the Gram–Schmidt process to obtain an orthonormal basis.

6. (Section 5.6, problems 5a and 6a)
   (a) Let
   \[
   A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}.
   \]
   Use the Gram–Schmidt process to find an orthonormal basis for the column space of \( A \).
   
   (b) Let
   \[
   B = \begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 2 \end{pmatrix}.
   \]
   Use the Gram–Schmidt process to find an orthonormal basis for the column space of \( B \).

7. (Section 5.6, problem 8) Use the Gram–Schmidt process to find an orthonormal basis for the subspace of \( \mathbb{R}^4 \) spanned by
   \[
   \mathbf{x}_1 = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}
   \]

2. Extra practice

These problems are suggested in case you would like extra practice with the concepts from class. They are not to be turned in.

- Section 5.5: Problems 6, 24, 35.
- Section 5.6: Problems 7, 14.