Second Exam
Math 308-503, Fall 2015
13 November 2015

1. This exam has 8 questions and 7 pages for a total of 50 points. Make sure you have all pages before you begin.

2. This is a 50 minute exam.

3. Put your name and UIN on the front of the exam. Be sure that the exam is stapled together when you turn it in. (See question 1.)

4. This is a closed-book exam. All calculators are prohibited.

5. Unless otherwise specified, you may use any method discussed in class to solve the problems. Show your work.

Name: __________________________
UIN: __________________________

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2. Fill in the blank with the appropriate response.

(a) (2 points) If a driven system is described by the differential equation $y'' + 9y = \sin(\omega t)$, what value of $\omega$ will maximize the response of the system? (In other words, what is the resonant frequency of the system?)

(a) 3

(b) (2 points) Suppose $A$ is a $4 \times 3$ matrix and $B$ is a $3 \times 4$ matrix. How many rows will the product $AB$ have?

(b) 4

(c) (2 points) Suppose $A$ is a $5 \times 2$ matrix and $B$ is a $2 \times 2$ matrix. How many columns will $AB$ have?

(c) 2

(d) (2 points) True or false: If $A$ and $B$ are both invertible $n \times n$ matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.

(d) T
3. (6 points) From the definition (i.e., without just writing down the answer from a table), compute the Laplace transform of the function $f(t) = te^t$.

**Solution:** The Laplace transform of $f$ is given by

$$
\int_0^\infty f(t)e^{-st} dt,
$$

so we can calculate using integration by parts (with $u = t$ and $v = \frac{1}{s-1}e^{-(s-1)t}$):

\[
\int_0^\infty te^t e^{-st} dt = \int_0^\infty te^{-(s-1)t} dt \\
= (\frac{-t}{s-1}e^{-(s-1)t}|_0^\infty + \frac{1}{s-1} \int_0^\infty e^{-(s-1)t} dt) \\
= 0 - \frac{1}{(s-1)^2}e^{-(s-1)t}|_0^\infty = \frac{1}{(s-1)^2}.
\]
4. This question asks you to find inverse Laplace transforms of two functions. (In other words, to find the function of $t$ whose Laplace transform is the given function of $s$.)

(a) (4 points) Find $L^{-1}\left\{\frac{4}{s^2+4}\right\}$.

**Solution:** You can find this on your table: The Laplace transform of $\sin(2t)$ is $\frac{2}{s^2+4}$, so the inverse Laplace transform of $\frac{4}{s^2+4}$ is $2\sin(2t)$.

(b) (5 points) Find $L^{-1}\left\{\frac{4}{s(s^2+4)}\right\}$.

**Solution:** You can either use partial fractions or the convolution theorem here. For partial fractions, you observe that

$$\frac{4}{s(s^2+4)} = \frac{1}{s} - \frac{s}{s^2 + 4}$$

and then use your table. To use the convolution theorem, you note that the inverse Laplace transform of $2/s$ is $2$, while the inverse Laplace transform of $2/(s^2 + 4)$ is $\sin(2t)$, so that the inverse Laplace transform of the product is

$$\int_0^t 2\sin(2\tau) \, d\tau = -\cos(2\tau)|_{\tau=0}^{\tau=t} = 1 - \cos(2t).$$
5. (7 points) Using any method you like, find the solution of the following initial value problem (feel free to use results you have already found on the test):

\[ y'' + 4y = 4 \]
\[ y(0) = 0 \]
\[ y'(0) = 4 \]

**Solution:** You can use what you’ve done above to make the Laplace transform method a bit easier. Take the Laplace transform of both sides to find (here \(Y(s)\) is the Laplace transform of \(y(t)\))

\[ s^2Y(s) - 4 + 4Y(s) = 4/s, \]

so that

\[ Y(s) = \frac{4}{s^2 + 4} + \frac{4}{s(s^2 + 4)}. \]

Using the inverse Laplace transforms you calculated, you end up with

\[ y(t) = 2 \sin(2t) + 1 - \cos(2t). \]

6. (3 points) Write the second order linear differential equation \(y'' + 6y' + 13y = 0\) as a first order linear system.

**Solution:** Introduce \(x_1 = y\) and \(x_2 = y'\). We then have the following system:

\[ x_1' = x_2 \]
\[ x_2' = -13x_1 - 6x_2 \]
7. (4 points) Suppose $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ are the Laplace transforms of $f$ and $g$, respectively. Write down an expression for the function whose Laplace transform is $F(s)G(s)$.

**Solution:** The inverse Laplace transform of a product is given by the convolution of the original two functions, so

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f \ast g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$ 

8. In this problem we consider the following first order linear system:

$$\mathbf{x}' = A\mathbf{x},$$

where $A$ is the $2 \times 2$ matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$ 

(a) (4 points) What are the eigenvalues of $A$?

**Solution:** We find when det$(A - \lambda I) = 0$, so when

$$0 = \text{det}(A-\lambda I) = \text{det}\left(\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}\right) = (1-\lambda)^2-4 = \lambda^2-2\lambda-3 = (\lambda-3)(\lambda+1),$$

so the two eigenvalues of $A$ are $\lambda = -1$ and $\lambda = 3$. 

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(b) (5 points) For each eigenvalue of $A$, find an eigenvector of $A$ with that eigenvalue. For your convenience, here is the matrix $A$ again:

\[
A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}
\]

**Solution:** Let’s start with $\lambda = -1$, so we want to find a vector $\mathbf{v}$ so that $(A + I)\mathbf{v} = 0$. We’ll row-reduce $A + I$:

\[
\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}
\]

so we need $2v_1 + 2v_2 = 0$. We can thus take $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as an eigenvector with eigenvalue $-1$.

For $\lambda = 3$, we row-reduce $A - 3I$:

\[
\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}
\]

so we need $-2v_1 + 2v_2 = 0$, so we can take $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as an eigenvector with eigenvalue 3.

(c) (3 points) Find two (linearly independent) solutions of the system $\mathbf{x}' = A\mathbf{x}$. (In other words, your two solutions should not be multiples of one another.)

**Solution:** The two solutions corresponding to the eigenvalues and eigenvectors we found above are:

\[
e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]