1. Consider the following three vectors in $\mathbb{R}^3$:

$$
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}
$$

Are the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent or linearly dependent?

Solution: We use Gaussian elimination to find out. We start with the matrix with these vectors as its columns:

$$
\begin{pmatrix}
1 & 2 & -1 \\
0 & 1 & 1 \\
1 & 3 & 1
\end{pmatrix}
$$

We subtract the first row from the third to obtain

$$
\begin{pmatrix}
1 & 2 & -1 \\
0 & 1 & 1 \\
0 & 1 & 2
\end{pmatrix}
$$

and then subtract the second row from the third and twice the second row from the first to yield

$$
\begin{pmatrix}
1 & 0 & -3 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

Finally we subtract the third row from the second and add three times the third row to the first to obtain

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

which is the identity matrix, so the three vectors are linearly independent.

2. (a) Find the eigenvalues and corresponding eigenvectors of the following matrix:

$$
A = \begin{pmatrix} 5 & 8 & 16 \\
4 & 1 & 8 \\
-4 & -4 & -11 \end{pmatrix}
$$
Solution: We start by finding the eigenvalues:

\[
\det(A - \lambda I) = \begin{vmatrix}
5 - \lambda & 8 & 16 \\
4 & 1 - \lambda & 8 \\
-4 & -4 & -11 - \lambda
\end{vmatrix}
\]

\[
= (5 - \lambda)((1 - \lambda)(-11 - \lambda) + 32) - 8 (4(-11 - \lambda) + 32) + 16(-16 + 4(1 - \lambda))
\]

\[
= -\lambda^3 - 5\lambda^2 - 3\lambda + 9 = -(\lambda - 1)(\lambda + 3)^2
\]

so the eigenvalues are 1 and -3. We now find the eigenvectors. First we consider \(\lambda = 1\):

\[
A - I = \begin{pmatrix}
4 & 8 & 16 \\
4 & 0 & 8 \\
-4 & -4 & -12
\end{pmatrix},
\]

Now we subtract the first row from the second and add the first row to the third:

\[
\begin{pmatrix}
4 & 8 & 16 \\
0 & -8 & -8 \\
0 & 4 & 4
\end{pmatrix}
\]

Continuing, we have

\[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

so that the \(\lambda = 1\) eigenvector can be taken to be

\[
\begin{pmatrix}
-2 \\
-1 \\
1
\end{pmatrix}
\]

Similarly, for the \(\lambda = -3\) eigenvectors, we reduce

\[
A + 3I = \begin{pmatrix}
8 & 8 & 16 \\
4 & 4 & 8 \\
-4 & -4 & -8
\end{pmatrix}
\]

which reduces to

\[
\begin{pmatrix}
1 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

This has two free columns, so we can pick two linearly independent eigenvectors here. We’ll do so by taking \(x_2 = 0\) and \(x_3 = 1\) and then swapping:

\[
\begin{pmatrix}
-2 \\
0 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
-1 \\
1 \\
0
\end{pmatrix}
\]

are both eigenvectors with eigenvalue -3.
(b) Use your answer from above to find three linearly independent solutions of the system of differential equations given by

\[ x_1' = 5x_1 + 8x_2 + 16x_3 \]
\[ x_2' = 4x_1 + x_2 + 8x_3 \]
\[ x_3' = -4x_1 - 4x_2 - 11x_3 \]

**Solution:** We can take the following three solutions:

\[ e^t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \quad e^{-3t} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad e^{-3t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \]