1. This exam has 4 questions and 6 pages for a total of 110 points. Make sure you have all pages before you begin.

2. As a form of “extra credit”, the grade will be counted as if it were out of 100 points.

3. This is a 50 minute exam.

4. Put your name and UIN on the front of the exam. Be sure that the exam is stapled together when you turn it in. (See question 1.)

5. This is a closed-book exam. All calculators are prohibited.

Name: 

UIN: 

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1. (2 points) Make sure that your name and UIN are on the front of the exam. Make sure that the exam is stapled together when it is turned in.

2. (36 points) (a) Alice is moving at speed $v$ in the positive $x$ direction relative to me. Write down the Lorentz transformation from my coordinate system to Alice’s. (Take $c = 1$.)

**Solution:** Let $\gamma_v = \frac{1}{\sqrt{1 - v^2}}$. The transformation has matrix

$$
\begin{pmatrix}
\gamma_v & -\gamma_v v & 0 & 0 \\
-\gamma_v v & \gamma_v & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(b) I observe a photon moving in the positive $x$ direction with energy $E$. What energy does Alice observe the photon as having?

**Solution:** I observe the photon as having energy $E$, so in my coordinate system, its momentum has components $\mathbf{p} \to (E, E, 0, 0)$. There are at least two ways to proceed from here: I can find that Alice’s 4-velocity has the following components in my coordinate system: $U_A \to (\gamma_v, \gamma_v v, 0, 0)$, so that the energy she observes is $-U_A \cdot \mathbf{p} = \gamma_v E (1 - v)$.

Alternatively, I can compute the components of $\mathbf{p}$ in Alice’s frame: $p^t' = \gamma E - v \gamma E$, $p^x' = -v \gamma E + \gamma E$, so that the energy is $\gamma E (1 - v)$.

(c) Bob is moving at speed $u$ in the positive $x$ direction relative to Alice. Suppressing the irrelevant $y$ and $z$ directions, find the Lorentz transformation from my coordinate system to Bob’s. (Multiply two $2 \times 2$ matrices.)

**Solution:** By analogy with the previous part, the transformation taking Alice’s coordinate system to Bob’s is given by

$$
\begin{pmatrix}
\gamma_u & -\gamma_u u \\
-\gamma_u u & \gamma_u
\end{pmatrix}
$$

where $\gamma_u = \frac{1}{\sqrt{1 - u^2}}$.

Thus, in order to get from my coordinate system to Bob’s, we have

$$
\begin{pmatrix}
\gamma_u & -\gamma_u u \\
-\gamma_u u & \gamma_u
\end{pmatrix}
\begin{pmatrix}
\gamma_v & -\gamma_v v \\
-\gamma_v v & \gamma_v
\end{pmatrix}
= \frac{1}{\sqrt{1 - u^2} \sqrt{1 - v^2}}
\begin{pmatrix}
1 + uv & -u - v \\
-u - v & 1 + uv
\end{pmatrix}
$$

(d) From your answer to the previous part, deduce the (one-dimensional) velocity composition law in special relativity (the formula for Bob’s speed relative to mine).
Solution: We must simplify the above to the form

\[
\frac{1}{\sqrt{1 - w^2}} \begin{pmatrix} 1 & -w \\ -w & 1 \end{pmatrix}
\]

Observe that \(-w\) must be the ratio of the off-diagonal and diagonal elements, so \(w = \frac{u+v}{1+uv}\). (One can easily check that the \(\gamma\) factor works out here.)
3. (36 points) Consider the coordinates $\sigma$ and $\lambda$ on 2-dimensional Minkowski space (flat spacetime) given by the following:

\[ t = \sigma \cosh \lambda, \quad x = \sigma \sinh \lambda, \quad 0 < \sigma < \infty, \quad -\infty < \lambda < \infty \]

(a) What region of the $(t, x)$-plane is covered by the $(\sigma, \lambda)$ coordinate system?

**Solution:** We have that $\sigma > 0$, so $t > 0$, while $x$ can have both signs. We also have that $t^2 - x^2 = \sigma^2 > 0$, so we must have $t > |x|$.

(b) Compute the basis vectors $e_\sigma$ and $e_\lambda$.

**Solution:** The inverse of the Jacobian of the coordinate change taking us from $(t, x)$ to $(\sigma, \lambda)$ is given by

\[
\Lambda^{-1} = \begin{pmatrix}
\frac{\partial t}{\partial \sigma} & \frac{\partial t}{\partial \lambda} \\
\frac{\partial x}{\partial \sigma} & \frac{\partial x}{\partial \lambda}
\end{pmatrix} = \begin{pmatrix}
cosh \lambda & \sigma \sinh \lambda \\
\sinh \lambda & \sigma \cosh \lambda
\end{pmatrix}
\]

with inverse

\[
\Lambda = \begin{pmatrix}
\frac{\partial \sigma}{\partial t} & \frac{\partial \sigma}{\partial \lambda} \\
\frac{\partial \lambda}{\partial t} & \frac{\partial \lambda}{\partial \lambda}
\end{pmatrix} = \begin{pmatrix}
cosh \lambda & -\sinh \lambda \\
-\frac{1}{\sigma} \sinh \lambda & \frac{1}{\sigma} \cosh \lambda
\end{pmatrix}
\]

$e_\sigma$ is then given by $(\Lambda^{-1})^\sigma_\alpha e_\alpha = \cosh \lambda e_t + \sinh \lambda e_x$. A similar computation gives

\[
e_\sigma = \cosh \lambda e_t + \sinh \lambda e_x,
\]

\[
e_\lambda = \sigma \sinh \lambda e_t + \sigma \cosh \lambda e_x.
\]

(c) Compute the dual basis $\tilde{\omega}^\sigma = d\sigma$ and $\tilde{\omega}^\lambda = d\lambda$ in terms of $dt$ and $dx$.

**Solution:** There are at least two ways to do this. One is to use the Jacobian change, so that

\[
\tilde{\omega}^\sigma = \Lambda^\sigma_\alpha dx + \Lambda^\alpha_\sigma dy = \cosh \lambda dt - \sinh \lambda dx,
\]

\[
\tilde{\omega}^\lambda = -\frac{1}{\sigma} \sinh \lambda dt + \frac{1}{\sigma} \cosh \lambda dx
\]

Another way is to observe that $\sigma^2 = t^2 - x^2$ and $\tanh \lambda = x/t$, so that

\[2\sigma \, d\sigma = 2t \, dt - 2x \, dx,
\]

\[\sech^2 \lambda \, d\lambda = -\frac{x}{t^2} \, dt + \frac{1}{t} \, dx \]

In other words,

\[
d\sigma = \frac{t}{\sigma} \, dt - \frac{x}{\sigma} \, dx = \cosh \lambda dt - \sinh \lambda dx
\]

\[
d\lambda = \cosh^2 \lambda \left( -\frac{\sinh \lambda}{\sigma \cosh^2 \lambda} \, dt + \frac{1}{\sigma \cosh \lambda} \, dx \right) = -\frac{1}{\sigma} \sinh \lambda dt + \frac{1}{\sigma} \cosh \lambda dx
\]
(d) Compute the Christoffel symbols associated to the coordinates \((\sigma, \lambda)\).

**Solution:** We compute \(\partial_\sigma e_\sigma\), \(\partial_\lambda e_\sigma\), \(\partial_\sigma e_\lambda\) and \(\partial_\lambda e_\lambda\):

\[
\partial_\sigma e_\sigma = 0
\]

\[
\partial_\lambda e_\sigma = \sinh \lambda e_t + \cosh \lambda e_x = \frac{1}{\sigma} e_\lambda
\]

\[
\partial_\sigma e_\lambda = \sinh \lambda e_t + \cosh \lambda e_x = \frac{1}{\sigma} e_\lambda
\]

\[
\partial_\lambda e_\lambda = \sigma \cosh \lambda e_t + \sigma \sinh \lambda e_x = \sigma e_\sigma,
\]

so that the following three Christoffel symbols are non-zero:

\[
\Gamma_\sigma^\lambda \lambda = \Gamma_\lambda^\sigma \lambda = \frac{1}{\sigma}, \quad \Gamma_\lambda^\sigma \lambda = \sigma.
\]

The remaining 5 symbols vanish.
4. (36 points) (a) Explain in modern language (with multilinear functions and so on) what a \((0,2)\)-tensor is.

**Solution:** A \((0,2)\)-tensor is a multilinear function of two vectors.

(b) A \((0,2)\)-tensor \(F\) is antisymmetric if \(F_{\beta\alpha} = -F_{\alpha\beta}\). Show that this property is independent of choice of basis.

**Solution:** Here is one way to do it: Suppose \(\Lambda^\alpha_{\mu'}\) is a Lorentz transformation. We then have

\[
F'_{\nu'\mu'} = \Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'}F_{\beta\alpha} = -\Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'}F_{\alpha\beta} = -F_{\mu'\nu'}.
\]

Another way is to note that \(F_{\alpha\beta} = F(e_\alpha, e_\beta)\), so that the antisymmetry condition and the fact that the \(e_\alpha\) form a basis imply that, for any vectors \(V\) and \(W\), we have

\[
F(V, W) = -F(W, V),
\]

which is clearly independent of choice of basis.

(c) Show that any \((0,2)\)-tensor can be written as the sum of a symmetric tensor (so that \(F_{\alpha\beta} = F_{\beta\alpha}\)) and an antisymmetric one.

**Solution:** Given \(F_{\alpha\beta}\), we write

\[
F_{\alpha\beta} = \frac{1}{2} (F_{\alpha\beta} + F_{\beta\alpha}) + \frac{1}{2} (F_{\alpha\beta} - F_{\beta\alpha}).
\]

The first term is symmetric and the second term is antisymmetric. (This is essentially the same construction as the odd and even parts of a function)

(d) How many independent components does a general \((0,2)\)-tensor have? An antisymmetric one? A symmetric one? (You may assume either 4 dimensions or \(n\) dimensions as you please.)

**Solution:** A general \((0,2)\) tensor has \(n^2\) independent components (\(n\) for each index).

An antisymmetric one must have \(F_{\alpha\beta} = -F_{\beta\alpha}\), which has two consequences. First, it shows that \(F_{\alpha\alpha} = 0\), so there are no independent components with only one distinct index. If the indices are distinct, you can use the antisymmetry condition to make sure that \(\alpha < \beta\), so you must only count the number of pairs \((\alpha, \beta)\), where \(\alpha\) and \(\beta\) have \(n\) choices but \(\alpha < \beta\). This sort of looks like a triangle and so has \(n(n - 1)/2\) independent components. (6 in dimension 4)

A similar argument works for the symmetric case, but you can only arrange that \(\alpha \leq \beta\), which gives you \(n\) more components and hence \(n(n + 1)/2\) independent
components (10 in dimension 4). Alternatively, you could just use part (c) and observe that in this case you must have $n^2 - n(n-1)/2 = n(n+1)/2$ components.