Abstract. An a posteriori error bound for the maximum (pointwise) error for the interior penalty discontinuous Galerkin method for a standard elliptic model problem on polyhedral domains is presented. The computational domain is not required to be Lipschitz, thus allowing for domains with cracks and other irregular polyhedral domains. The proof is based on direct use of Green's functions and varies substantially from the approach used in previous proofs of similar $L^\infty$ estimates for (continuous) finite element methods in the literature. Numerical experiments indicating the good behaviour of the resulting a posteriori bounds within an adaptive algorithm are also presented.

1. Introduction. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygonal ($d = 2$) or polyhedral ($d = 3$) domain. We consider the Poisson problem
\begin{equation}
-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega,
\end{equation}
with $f \in L^\infty(\Omega)$. We assume that the boundary data $g \in C^0(\partial \Omega)$ and that $g$ can be extended to $\Omega$ so that $g \in W^{1,q}(\Omega)$ for some $q > d$, where $W^{k,p}(\Omega)$ denotes the standard Sobolev space or order $k$ based on the Lebesgue space $L_q(\Omega)$. We emphasize that we only require that $\Omega$ be polyhedral, and in particular do not impose that $\Omega$ has a Lipschitz boundary. Well-known non-Lipschitz polyhedral domains include domains with cracks, that is, domains where interior(\Omega) \ \Omega has codimension 1. Other examples include the three-dimensional “two-brick” domain which we describe more thoroughly below (cf. [43]). Note also that extension to elliptic problems with coefficients is possible so long as certain regularity conditions are satisfied; we do not pursue this in any detail as our main interest is in establishing results that admit a sufficiently broad class of domains.

Computable a posteriori error estimates are an important part of computational practice because of their ability to provide computable information about errors and drive adaptive mesh refinement algorithms. There has been a substantial activity in recent years for the derivation of a posteriori bounds for discontinuous Galerkin (IPDG) methods for elliptic problems (see, e.g., [34], [7], [2], [35], [33], [11], [27], [9], [26], [45], [21] and the references therein). Our goal in this work is to prove a posteriori error estimates for the maximum norm ($L^\infty$) error for interior penalty discontinuous Galerkin (IPDG) methods for (1.1). Controlling the error in the maximum norm is of natural interest in some applications, especially in certain types of nonlinear problems (cf. [28], [40], [11], [39]). For some nonlinear problems maximum-norm estimates play an important role in establishing rigorous error estimates even in integral norms; cf. [7], where maximum-norm estimates for the (linear stationary) Poisson problem are used to prove $L^\infty(L_2)$ bounds for the (semilinear parabolic) Allen-Cahn equation.

Pointwise or maximum-norm error control is relatively difficult to carry out rigorously because of the variational nature of the finite element method. In the context of stationary linear problems, such error estimates have previously been considered in...
These are most relevant to our technical development, and we describe them in more detail below. Extensions of these works include obstacle and semilinear elliptic problems ([28], [40], [41], [39]) and parabolic problems ([25], [8], [18], [20]). All of the above works consider standard continuous Galerkin spatial discretizations; [14] also includes estimates for Crouzeix-Raviart nonconforming elements.

We next briefly describe our main result. Let $u_h$ be an IPDG approximation to $u$ on a quadrilateral or simplicial decomposition $\mathcal{T}$ of $\Omega$. The mesh is allowed to be nonconforming, but with a bounded depth of nonconformity. Let also $[v]$ denote the jump in $v$ across an element interface if $v$ is a scalar quantity, or the jump in the normal component of $v$ if $v$ is a vector quantity. Let also $h$ denote the local mesh size and $\Gamma_{\text{int}}$ the set of interior edges of $\mathcal{T}$ and denote by $\Delta_h$ the elementwise Laplacian. Then,

\[
\|u - u_h\|_{L_\infty(\Omega)} \leq C \ell_{h,d}(h^2(f + \Delta_h u_h)\|_{L_\infty(\Omega)} + \|h[\nabla u_h]\|_{L_\infty(\Gamma_{\text{int}})} + \|u_h\|_{L_\infty(\Gamma_{\text{int}})} + \|g - u_h\|_{L_\infty(\partial \Omega))}).
\]  

(1.2)

Here $\ell_{h,d} = (\ln 1/h)^{\alpha_d}$, where $h = \min_\Omega h$, $\alpha_2 = 2$, and $\alpha_3 = 1$. We restate (1.2) more precisely in Theorem 5.1 below.

A main challenge associated with proving $L_\infty$ error bounds is the effective representation of pointwise errors. All of the above-cited works concerning $L_\infty$ error control for continuous Galerkin methods employ either the standard (continuous) Green’s function for this purpose or a regularized Green’s function. In the context of linear elliptic problems, [38] employs a regularized Green’s function to prove a posteriori estimates for polygonal domains in $\mathbb{R}^2$. This leads to the presence of a “regularization penalty” having a priori character in the upper bound, which in [38] is removed only under a nondegeneracy assumption on the mesh which can compromise optimality properties if enforced (cf. [19]). Similar estimates for $d = 3$ and estimates for Crouzeix-Raviart nonconforming elements are found in [14]; both of these works assume $g = 0$. This approach was extended to semilinear problems with nonhomogeneous Dirichlet data in [39], where it was also shown that the “regularization penalty” may be bounded by the other terms in the a posteriori upper bound if the domain is Lipschitz. [24] employs the standard Green’s function as we do here in order to obtain maximum-norm a posteriori estimates for polygonal domains in $\mathbb{R}^2$. The proofs use a decomposition of the Green’s function into singular functions that separately account for corner singularities and the singular portion of the fundamental solution. These ideas do not easily extend to $\mathbb{R}^3$ since vertex and edge singularities are difficult to write down precisely in this context and, thus, we do not assume such a decomposition in this work.

In order both to account for additional technicalities associated with discontinuous Galerkin methods and in order to obtain the most general possible results, we have developed an approach that varies substantially in several technical details from the previous works cited above. As in [24], we employ the standard Green’s function for (1.1), but we employ analytical properties of the Green’s function rather differently. Our proofs corresponding lead to several improvements in the final result even for continuous Galerkin methods. In contrast to [38], [14], and [24], we allow nonhomogeneous Dirichlet boundary data, while also giving a rigorous proof for both $d = 2$ and $d = 3$ and avoiding any a priori terms or nondegeneracy assumptions in our results. While [39] likewise is valid for nonzero boundary data, for space dimensions $d = 2, 3$, and avoids the regularization penalty, the results in that work are not valid
for non-Lipschitz polyhedral domains as ours are. In addition, the techniques used in [39] to remove the regularity penalty do not seem to be directly applicable to discontinuous Galerkin methods as they in essence involve Hölder regularity bounds for the finite element error $u - u_h$. In the case $d = 3$ we also improve upon the exponent in the logarithmic factor present in the upper bound. In particular, the proof of [14] (cf. [39]) leads to a logarithmic factor $(\ln(1/h))^{4/3}$, whereas our logarithmic factor for $d = 3$ is simply $\ln(1/h)$.

We finally make some comments concerning domain regularity. As emphasized above, we admit non-Lipschitz polyhedral domains in our analysis, while some previously published similar results have relied on regularity estimates that only appear in the literature for Lipschitz domains. The two-brick example in Section 7 demonstrates why extension to general polyhedral domains is desirable. As is standard on polyhedral domains, solutions to (1.1) on the two-brick domain may be decomposed into a regular portion, edge singularities, and vertex singularities. These singularities can be characterized by solving an eigenvalue problem for the Laplace-Beltrami operator on a spherical subdomain. The non-Lipschitz nature of the two-brick domain is due to a vertex at which various limits of normals to $\partial \Omega$ point in opposite directions, depending on the direction from which the limit is taken. Using an adaptive eigenvalue solver for the Laplace-Beltrami problem, we characterize below the singularity at this non-Lipschitz vertex and show that it has little effect on the regularity of the solution to (1.1). Rather, it is certain edge singularities which are not related to the non-Lipschitz nature of the domain that most strongly limit solution regularity and convergence rates of the adaptive method.

The rest of this work is structured as follows. In §2 we present a number of relevant regularity results for the Green’s function of the problem (1.1). In §3, the interior penalty discontinuous Galerkin method is introduced. Some useful stability and approximation estimates for local projections are presented in §4. The a posteriori error bound is then proven in §5 and its efficiency is shown. In §6 we state and prove an a posteriori $L_\infty$ error bound for continuous Galerkin methods which improves on similar bounds in the literature. We conclude with some two- and three- dimensional numerical examples in §7, indicating the good performance of the derived a posteriori error estimates within an adaptive algorithm.

2. Green’s function and regularity. We collect a number of essential properties of Green’s functions for the model problem (1.1) and also give regularity results.

For $\omega \subset \mathbb{R}^d$, $d = 2, 3$, let $W^{k,p}(\omega)$, $k \geq 0$, $p \in [1, \infty]$ signify the usual Sobolev spaces equipped with the norm $\| \cdot \|_{W^{k,p}(\omega)}$ and seminorm $| \cdot |_{W^{k,p}(\omega)}$; for $k = 0$, we retrieve the standard Lebesgue spaces $L_p(\omega)$, with corresponding norms $\| \cdot \|_{L_p(\omega)}$. We shall consider the spaces $W^{k,p}_0(\omega)$, defined for $p < \infty$ as the completion of $C_c^\infty(\omega)$ in the $\| \cdot \|_{W^{k,p}(\omega)}$-norm. Finally, we shall use the standard notation $H^s(\omega) := W^{s,2}(\omega)$ and $H_0^s(\omega) := W_0^{s,2}(\omega)$.

We begin by giving some regularity results.

**Lemma 2.1.** Assume that $f \in L_p(\Omega)$ for some $1 < p < \frac{4}{3}$ and that $g = 0$. Then there is a unique solution $u \in W^{2,p}(\Omega)$ to (1.1), and

$$\|u\|_{W^{2,p}(\Omega)} \leq C_p\|f\|_{L_p(\Omega)}.$$  \hspace{1cm} (2.1)

In addition, $u \in W^{1,q}_0(\Omega)$ for some $q > d$.

**Proof.** (2.1) may be found in [30] and [15] when $d = 2$. In the case $d = 3$, see [16] (Corollary 3.10) and [37] (Theorem 4.3.2 and comments following; the latter reference
The correction introduced in order to guarantee that the principle then yields the logarithmic bound in (2.6).

Also, for any fixed $1 \leq q < \frac{d}{d-1}$,

$$\|G(\cdot, y)\|_{W^{1,q}(\Omega)} \leq C,$$

(2.5)

where $C$ depends on $\Omega$ but is independent of $y$. Finally, for $(x, y) \in \Omega \times \Omega$, $x \neq y$, we have

$$|G(x, y)| \leq \begin{cases} C|x-y|^{-1}, & d = 3, \\ C\ln \frac{C}{|x-y|}, & d = 2. \end{cases}$$

(2.6)

Proof. For $d = 3$, the fact that $G(\cdot, y) \in W^{1,q}_0(\Omega) \cap H^1(\Omega \setminus \overline{B_r(y)})$, (2.5), and (2.6) are directly contained in [31]. $\Box$

For $d = 2$, $G(\cdot, y) \in W^{1,q}_0(\Omega) \cap H^1(\Omega \setminus \overline{B_r(y)})$, (2.4) (after a density argument as above), and (2.5) are found in [23]. The remaining assertion $G(\cdot, y) \in W^{2,p}(\Omega \setminus B_r(y))$ in (2.2) and (2.3) then follow as in the case $d = 3$. The pointwise bound (2.6) is contained in [22] in the case of Lipschitz domains only. Note however that most of the references cited above are valid for general systems of elliptic equations, where maximum principles may be lacking. In the simpler current case of the scalar Laplacian, (2.6) for $d = 2$ may be obtained for general bounded domains using a straightforward argument. In particular, we use the representation $G(x, y) = -\frac{1}{2\pi} \ln |x-y| + \Phi(x, y)$, where $-\frac{1}{2\pi} \ln |x-y|$ is the fundamental solution in $\mathbb{R}^2$ and $\Phi(x, y)$ is a harmonic correction introduced in order to guarantee that $G(x, y) = 0$ for $x \in \partial\Omega$. The maximum principle then yields the logarithmic bound in (2.6). $\Box$
We next develop Green’s functions identities for nonhomogeneous Dirichlet boundary data. Before doing so, we pause to note that standard arguments justifying integration by parts are valid only for Lipschitz domains. However, integration by parts may be justified for general polyhedral domains by integrating by parts elementwise over a triangulation of \(\Omega\), then canceling terms on interior triangle boundaries (cf. [43], p. 503). We may thus integrate by parts on the domains considered here, as we do in the proof of the following lemma.

**Lemma 2.3.** Assume that boundary data \(g\) is given with \(g \in W^{1,q}(\Omega)\) for some \(q > d\). Assume also without loss of generality that \(g\) is harmonic in \(\Omega\). Then for \(y \in \Omega\), we have

\[
g(y) = -\int_{\partial\Omega} g(x) \partial_n G(x, y) \, ds,
\]

with \(\partial_n := \frac{\partial}{\partial n}\) denoting the derivative in the normal direction to the boundary. In addition,

\[
\|g\|_{L_\infty(\Omega)} \leq \|g\|_{L_\infty(\partial\Omega)}.
\]

**Proof.** Let \(y \in \Omega\) be fixed but arbitrary. Let also \(\omega \in C_0^\infty(\Omega)\) be a cutoff function that is 1 on a ball lying in \(\Omega\) and centered at \(y\). It is always possible to define such an \(\omega\) since \(y\) lies on the interior of \(\Omega\). Note in addition that the integral (2.7) is well defined, as a trace theorem (possibly applied element-wise on a triangulation of \(\Omega\) if \(\Omega\) is not Lipschitz) and (2.2) yield \(\partial_n G(\cdot, y) \in L^p(\partial\Omega)\) for \(1 \leq p < \frac{4}{3}\). Employing the \(W^{2, p}\) regularity of \(G\) on \(\text{supp}(1 - \omega)\), we integrate by parts and then note that \(G\) is harmonic on \(\text{supp}(1 - \omega)\) to compute

\[
\int_{\partial\Omega} g(x) \partial_n G(x, y) \, ds = \int_{\partial\Omega} (1 - \omega(x)) g(x) \partial_n G(x, y) \, ds
= \int_{\Omega} \Delta G(x, y) (1 - \omega(x)) g(x) \, dx + \int_{\Omega} \nabla G(x, y) \cdot \nabla [(1 - \omega)g](x) \, dx
= -\int_{\Omega} \nabla G(x, y) \cdot \nabla [\omega g](x) \, dx + \int_{\Omega} \nabla G(x, y) \nabla g(x) \, dx.
\]

The second integral in the last line of (2.9) disappears because \(g\) is harmonic in \(\Omega\), and by recalling that \(\omega(y) = 1\) and employing the representation (2.4) we have

\[
\int_{\Omega} \nabla G(x, y) \nabla [\omega g](x) \, dx = (\omega g)(y) = g(y).
\]

This completes the proof of (2.7).

The maximum principle (2.8) is standard under the regularity assumptions made here; cf. Theorem 8.1 of [29].

**3. Discontinuous Galerkin method.** Let \(T\) be a shape-regular subdivision of \(\Omega\) into disjoint open simplicial, quadrilateral \((d = 2)\), or hexahedral \((d = 3)\) elements \(\kappa\) so that \(\Omega = \cup_{\kappa \in T} \kappa\) and set \(h_\kappa := \text{diam } \kappa\). Letting \(\hat{\kappa}\) represent the unit reference element (either a box or simplex, as appropriate), we assume that for each element \(\kappa \in T\) there is an affine map \(A\) such that \(\kappa = A(\hat{\kappa})\). By shape-regular, we mean more precisely that \(|\nabla A| \lesssim h_\kappa\) and \(|\nabla A^{-1}| \lesssim h_\kappa^{-1}\). Note that if quadrilateral or hexahedral elements are used, the assumption of affine equivalence restricts us to considering
meshes consisting only of parallelograms ($d = 2$) or parallelepipeds ($d = 3$). Also, here and below we write $a \lesssim b$ when $a \leq C b$ for a constant $C$ depending possibly on the shape-regularity of $T$, geometric properties of $\Omega$, the depth of nonconforming refinement in the mesh, and the local polynomial degree $r$ of the finite element basis (defined below), but not on the local mesh size of $T$ or other essential quantities for the arguments below. Let $\Gamma$ be the union of all $(d - 1)$-dimensional element faces $e$ associated with the subdivision $T$ (including $\partial \Omega$). Let also $\Gamma_{\text{int}} := \Gamma \setminus \partial \Omega$, so that $\Gamma = \partial \Omega \cup \Gamma_{\text{int}}$.

We further assume that $T$ is derived by systematic refinement of an initial conforming mesh $T_0$, i.e., by bisection or red-green refinement of simplices or by quad refinement of hexahedra. We allow hanging nodes in our refinement procedures with the standard restriction that there exists an integer $M \geq 0$ such that no edge of $T$ contains more than $M$ hanging nodes. $T$ is then locally quasi-uniform, that is, given $\kappa, \kappa' \in T$ with $\kappa \cap \kappa' \neq \emptyset$, $h_{\kappa} \sim h_{\kappa'}$. In addition, the coarsest conforming refinement of $T$ lies at a bounded refinement depth from $T$. More precisely, there is an integer $M \geq 0$ (depending on $M$) such that a conforming simplicial refinement $\tilde{T}$ of $T$ exists in which no element of $\tilde{T}$ is subdivided more than $M$ times in passing from $T$ to $\tilde{T}$. The existence of $\tilde{T}$ follows easily from the fact that $T$ is derived by systematic refinement from a conforming parent mesh and its local quasi-uniformity. Note also that

$$\text{if } \kappa \in T \text{ and } \tilde{\kappa} \in \tilde{T} \text{ with } \tilde{\kappa} \subset \kappa, \text{ then } h_{\kappa} \sim h_{\tilde{\kappa}}.$$ (3.1)

Remark 3.1. In [9], quasi-optimality in the energy norm of an adaptive IPDG method is proved under a somewhat more restrictive assumption on the nonconformity depth. In particular, there it is in essence assumed that the finest conforming coarsening of $T$ has local mesh size equivalent to $T$, whereas we essentially assume local mesh size equivalence for the coarsest conforming simplicial refinement. The condition we impose is much less restrictive since as per the examples in [9], bounding the number of hanging nodes per edge does not guarantee that there is a bounded refinement depth between $T$ and its finest conforming coarsening.

At the other extreme, [3] contains energy-norm a posteriori bounds that are valid without the assumption that the number of hanging nodes per edge is bounded. The techniques used in that work rely on a regular (Helmholtz) decomposition of the solution and do not seem easily applicable to the case of $L_\infty$ error bounds.

For a nonnegative integer $r$, we denote by $\mathcal{P}_r(\kappa)$, the set of all polynomials on $\kappa$ of degree at most $r$ if $\kappa$ is a $d$-simplex, or of degree at most $r$ in each variable if $\kappa$ is a quadrilateral. Consider the discontinuous finite element space

$$V_\kappa := \{ v \in L^p(\Omega) : v|_\kappa \in \mathcal{P}_r(\kappa), \kappa \in T \}.$$ (3.2)

Let $\kappa^+, \kappa^-$ be two generic elements sharing a face $e := \kappa^+ \cap \kappa^- \subset \Gamma_{\text{int}}$ with respective outward normal unit vectors $\mathbf{n}^+$ and $\mathbf{n}^-$ on $e$. For $q : \Omega \rightarrow \mathbb{R}$ and $\phi : \Omega \rightarrow \mathbb{R}^d$, let $q^\pm := q|_{\kappa^+ \cup \kappa^-}$ and $\phi^\pm := \phi|_{\kappa^+ \cup \kappa^-}$, and set

$$\{q\}_e := \frac{1}{2} (q^+ + q^-), \quad \{\phi\}_e := \frac{1}{2} (\phi^+ + \phi^-),$$

$$[q]_e := q^+ \mathbf{n}^+ + q^- \mathbf{n}^-, \quad [\phi]_e := \phi^+ \cdot \mathbf{n}^+ + \phi^- \cdot \mathbf{n}^-;$$

if $e \subset \partial \kappa \cap \partial \Omega$, we set $\{\phi\}_e := \phi^+$ and $[q]_e := q^+ \mathbf{n}^+$. 


We also introduce the meshsize \( h : \Omega \to \mathbb{R} \), defined by \( h(x) = \text{diam } \kappa \), if \( x \in \kappa \setminus \partial \kappa \) and \( h(x) = \{ h \} \), if \( x \in \Gamma \).

Consider the bilinear form \( B(\cdot, \cdot) : V_h \times V_h \to \mathbb{R} \), defined by

\[
B(w, v) := \int_{\Omega} \nabla_h w \cdot \nabla_h v \, dx - \int_{\Gamma} \{ \nabla v \} \cdot [w] + \{ \nabla w \} \cdot [v] - \sigma [w] \cdot [v] \, ds, \tag{3.3}
\]

with the function \( \sigma : \Gamma \to \mathbb{R}_+ \) defined piecewise by \( \sigma|_{e} := C_{\sigma} h_e^2 / (h|_{e}) \), for some constant \( C_{\sigma} > 0 \). The interior penalty discontinuous Galerkin (IPDG) method then reads: find \( u_h \in V_h \) such that

\[
B(u_h, v_h) = \int_{\Gamma} f v_h \, ds + \int_{\partial \Omega} g(\nabla v_h \cdot \vec{n} + \sigma v_h) \, ds =: l(v_h), \quad \text{for all } v_h \in V_h. \tag{3.4}
\]

The corresponding energy norm \( \| \cdot \| \) is defined by

\[
\| w \| := \left( \| \nabla_h w \|_{L^2(\Omega)}^2 + \| \sqrt{\sigma} [w] \|_{L^2(\Gamma)}^2 \right)^{1/2}
\]

for \( w|_{\kappa} \in W^{1,p}(\kappa) \), \( \kappa \in T \). The IPDG method is coercive in \( V_h \) with respect to this norm when \( C_{\sigma} \) is chosen sufficiently large (see, e.g., [4] for details).

We extend the definition of (3.3) to \( (H^1_0(\Omega) + V_h) \times (H^1_0(\Omega) + V_h) \), by

\[
B(w, v) := \int_{\Omega} \nabla_h w \cdot \nabla_h v \, dx
- \int_{\Gamma} \{ \nabla \pi v \} \cdot [w] + \{ \nabla v \} \cdot [\pi] - \sigma [w] \cdot [v] \, ds, \tag{3.5}
\]

where \( \pi : L^1(\Omega) \to V_h \) is some projection operator onto \( V_h \) (to be made precise later). Notice that, since \( \pi v = v \) for \( v \in V_h \), (3.5) is indeed an extension of (3.3).

4. Projective Operators. Here we give a precise definition for the projection \( \pi : L^1(\Omega) \to V_h \) which is convenient for the a posteriori analysis below. The idea for the definition of this projection is inspired by the classical Clement [13] or Scott-Zhang [12] averaging operators. Here, however, the projections admit completely local stability and approximation bounds as the finite element spaces are discontinuous (cf. also the discussion after Corollary 4.1 in [12]).

For each element \( \kappa \in T \), we consider its Lagrange basis \( \{ \phi_i^{\kappa} \}_{i=1}^{n_r} \), with \( n_r \) the number of elemental degrees of freedom, depending on the type of element and on the polynomial degree \( r \). We define \( \{ \psi_i^{\kappa} \}_{i=1}^{n_r} \) to be the \( L^2(\kappa) \)-dual basis to \( \{ \phi_i^{\kappa} \}_{i=1}^{n_r} \), viz.,

\[
\int_{\kappa} \phi_i^{\kappa}(x) \psi_j^{\kappa}(x) \, dx = \delta_{ij}, \quad i,j \in \{1, \ldots, n_r\}, \tag{4.1}
\]

with \( \delta_{ij} \) denoting the standard Kronecker delta. For \( v \in L^1(\kappa) \), we define the projection \( \pi^\kappa : L^1(\kappa) \to \mathcal{P}_r(\kappa) \) by

\[
\pi^\kappa v(x) := \sum_{i=1}^{n_r} \phi_i^{\kappa}(x) \int_{\kappa} \psi_i^{\kappa}(\xi)v(\xi) \, d\xi.
\]

Note that we have \( \pi^\kappa \phi_i^{\kappa} = \phi_i^{\kappa} \) from (4.1) and, therefore \( \pi^\kappa \) is indeed a projection. We define \( \pi v|_\kappa := \pi^\kappa v \), for \( \kappa \in T \).
The stability and approximation properties of $\pi$ are given in the next lemma. These properties are standard, but we are not aware of the precise result that we need, so we sketch the proof.

**Lemma 4.1.** Let $\kappa \in T$, $\pi^\kappa : L^1(\kappa) \rightarrow \mathcal{P}_r(\kappa)$, let $s$ be an integer with $0 \leq s \leq r+1$, and let $v \in W^{s,1}(\kappa)$. Then

$$|\pi^\kappa v|_{W^{s,1}(\kappa)} \lesssim |v|_{W^{s,1}(\kappa)}, \quad (4.2)$$

and

$$\sum_{k=0}^s h_{\kappa}^{k-s}|v - \pi^\kappa v|_{W^{s,1}(\kappa)} \lesssim |v|_{W^{s,1}(\kappa)}. \quad (4.3)$$

**Proof.** An elementary scaling argument yields $\|\psi_i^\kappa\|_{L^\infty(\kappa)} \lesssim \kappa^{-d}$ and $|\phi_i^\kappa|_{W^{s,1}(\kappa)} \lesssim \kappa^{-d-s}$. Thus we have for any polynomial $p$ of degree less than $s$

$$|\pi^\kappa v|_{W^{s,1}(\kappa)} = |\pi^\kappa (v - p)|_{W^{s,1}(\kappa)} \lesssim \sum_{i=1}^n |\phi_i^\kappa|_{W^{s,1}(\kappa)} \int_{\kappa} \psi_i^\kappa(\xi)(v - p)(\xi)d\xi \lesssim h_{\kappa}^{d-s} \sum_{i=1}^n \|\psi_i^\kappa\|_{L^\infty(\kappa)} \|v - p\|_{L^1(\kappa)} \lesssim n_r h_{\kappa}^{-d} \|v - p\|_{L^1(\kappa)} \lesssim |v|_{W^{s,1}(\kappa)},$$

where in the last step we have chosen $p$ so as to minimize $\|v - p\|_{L^1(\kappa)}$ over all polynomials of degree $s-1$ and then applied the scaled Bramble-Hilbert estimate

$$\inf_{p \in \mathcal{P}_{s-1}(\kappa)} \|v - p\|_{W^{s,1}(\kappa)} \lesssim h_{\kappa}^{s-k} \|v\|_{W^{s,1}(\kappa)}, \quad 0 \leq k \leq s$$

(cf. Theorem 4.3.8 of [10]). Given the stability estimate (4.2), the approximation bound similarly follows from the Bramble-Hilbert Lemma.

We also need to estimate the distance from a function $v_h \in V_h$ to a suitable conforming function $\chi \in H^1(\Omega)$ by the size of the jumps of $v_h$. This is the content of the next lemma (cf. [3] and [1]).

**Lemma 4.2.** Let $\tilde{T}$ be the minimal conforming refinement of $T$ described in [3]. For every $v_h \in V_h$, there exists a function $\chi \in W^{1,\infty}(\Omega)$ which is a piecewise polynomial on $\tilde{T}$ such that

$$\|v_h - \chi\|_{L^\infty(\Omega)} \lesssim \|[v_h]\|_{L^\infty(\Gamma_{int})} + \|g - v_h\|_{L^\infty(\partial\Omega)}. \quad (4.4)$$

**Proof.** Let $\tilde{V}_h$ be the corresponding to $\tilde{T}$ conforming finite element space of degree $r$ if $T$ is simplicial, so that the corresponding refined mesh $\tilde{T}$ lies at a uniformly bounded refinement depth from $T$. Recall that the conforming refinement $\tilde{T}$ is simplicial even if $T$ is quadrilateral or hexahedral. We thus let $\tilde{V}_h$ include simplicial elements of degree $2r$ in that case in order to ensure that $V_h|_{\tilde{k}} \subset \tilde{V}_h|_{\tilde{k}}$ for all $\tilde{k} \subset \tilde{T}$. Let also $I_{\tilde{T}}$ be the set of all Lagrange nodes of $\tilde{T}$, $I_{\tilde{T},\text{int}}$ be the set of interior Lagrange nodes, and $I_{\tilde{T},\partial\Omega} = I_{\tilde{T}} \setminus I_{\tilde{T},\text{int}}$.

Let $z \in I_{\tilde{T}}$ be a Lagrange node and let $\psi_z$ be the corresponding Lagrange basis function. In addition, let $P_z = \{\kappa \in T : z \in \pi\}$. Given $z \in I_{\tilde{T},\text{int}}$, let $N_z = \#\{\kappa \in T : \}$
\( z \in \mathcal{P} \). We then define
\[
\chi_z = \begin{cases} 
\frac{1}{N_z} \sum_{k \in P_z} v_{h,k'}(z), & z \in I_{T,\text{int}}, \\
g(z), & z \in I_{T,\partial\Omega}.
\end{cases}
\tag{4.5}
\]

Here \( v_{h,k'}(z) \) is the value that \( v_h|_{\kappa'} \) takes on at \( z \). Also, let
\[
\chi(x) = \sum_{z \in I_T} \chi_z \psi_z(x).
\tag{4.6}
\]

A standard property of the Lagrange interpolant is that \( \|\psi_z\|_{L_\infty(\Omega)} \leq C \) (with \( C = 1 \) when the polynomial degree \( r = 1 \)). Let now \( I_{T,\Gamma} = \{ z \in I_T : z \in \Gamma \} \). Note that for \( z \in I_T \setminus I_{T,\Gamma} \), we have \( v_h(z) = \chi(z) \). Since \( v_h \) is a polynomial of degree \( r \) on any \( T \in \mathcal{T} \) in the case of simplicial elements or of degree \( 2r \) in the case of hexahedral elements, we have for any such \( T \) that
\[
\|v_h - \chi\|_{L_\infty(\Omega)} = \|\sum_{z \in I_T} (v_h(z) - \chi(z))\|_{L_\infty(T)} \\
\lesssim \sum_{z \in I_T \setminus I_{T,\partial\Omega}} \frac{1}{N_z} \sum_{k \in P_z} |v_{h,k}(z) - v_{h,k'}(z)| + \sum_{z \in \mathcal{P}_z \setminus I_{T,\partial\Omega}} |v_{h,k}(z) - g(z)| \\
\lesssim \|v_h\|_{L_\infty(\partial\Omega \setminus \partial\Omega)} + \|v_h - g\|_{L_\infty(\partial\Omega' \setminus \partial\Omega)}.
\tag{4.7}
\]

Thus, as desired,
\[
\|v_h - \chi\|_{L_\infty(\Omega)} \lesssim \|v_h\|_{L_\infty(\Gamma_{\text{int}})} + \|g - v_h\|_{L_\infty(\partial\Omega)}. \tag{4.8}
\]

\[\square\]

5. A posteriori error bounds. In this section we state and prove reliability and efficiency results.

5.1. Reliability. We are now ready to state and prove the main theorem.

Theorem 5.1. Let \( \Omega \subset \mathbb{R}^d \), \( d = 2,3 \), be a bounded polyhedral domain, and assume that \( u \) solves (1.1) with data \( f \in L_\infty(\Omega) \), \( g \in C(\partial\Omega) \cap W^{1,q}(\Omega) \) for some \( q > d \). Assume also that the number of hanging nodes per edge of elements in \( T \) is uniformly bounded. Then the following error bound holds:
\[
\|u - u_h\|_{L_\infty(\Omega)} \lesssim \ell_{h,d} \left( \|h^2(f + \Delta_h u_h)\|_{L_\infty(\Omega)} + \|h [\nabla u_h]\|_{L_\infty(\Gamma_{\text{int}})} \\
+ C_{\sigma, r^2} (\|u_h\|_{L_\infty(\Gamma_{\text{int}})} + \|g - u_h\|_{L_\infty(\partial\Omega)}) \right),
\tag{5.1}
\]
where \( \ell_{h,d} = 1 + \lfloor \ln(1/h) \rfloor \alpha^d \). Here \( h = \min_{\kappa \in \mathcal{T}} h_{\kappa} \), \( \alpha_2 = 2 \), and \( \alpha_3 = 1 \).

Proof. Let \( x_0 \in \Omega \setminus \Gamma \); then there is an element \( \kappa \in \mathcal{T} \) such that \( x_0 \in \kappa \). Letting \( G = G(\cdot, x_0) \) be the Green’s function as above with singularity at \( x_0 \), we have from (2.4) and (2.7) that
\[
u(x_0) = \int_{\Omega} \nabla u \cdot \nabla G dx - \int_{\partial\Omega} g \partial_{\Omega} G ds \under our assumptions.
Employing (3.4) while recalling that \( G = 0 \) on \( \partial \Omega \), we thus compute

\[
(u - u_h)(x_0) = u(x_0) - u_h(x_0)
\]

\[
= \int_\Omega \nabla u \cdot \nabla G \, dx - \int_{\partial \Omega} g \partial_n G \, ds - u_h(x_0)
\]

\[
= \int_\Omega fG \, dx - \int_{\partial \Omega} g \partial_n G \, ds - B(u_h, G) + B(u_h, G) - u_h(x_0)
\]

\[
= \int_\Omega fG \, dx - \int_{\partial \Omega} g \partial_n G \, ds - l(\pi G) - B(u_h, G - \pi G)
\]

\[
+ B(u_h, G - u_h(x_0))
\]

\[
= \left[ \int_\Omega f(G - \pi G) \, dx - B(u_h, G - \pi G) + \int_{\partial \Omega} \sigma G - \pi G \, ds \right]
\]

\[
+ \left[ - \int_{\partial \Omega} g \partial_n (\pi G) \, ds + B(u_h, G) - u_h(x_0) \right] =: [I] + [II].
\]

Next we note the formula

\[
\sum_{\kappa \in T_h} \int_{\partial \kappa} v \cdot \vec{n} \, ds = \int_{\Gamma} [v] \cdot \{q\} \, ds + \int_{\Gamma_{\text{int}}} [v][q] \, ds
\]

(5.3)

for functions \( v : \Omega \to \mathbb{R} \) and \( q : \Omega \to \mathbb{R}^d \), that may be discontinuous across \( \Gamma_{\text{int}} \). In order to bound the term \( I \), we employ (3.5) while noting that \( \pi(G - \pi G) = 0 \), integrate by parts, use (5.3), and rearrange terms to find

\[
I = \int_\Omega f(G - \pi G) \, dx - \int_\Omega \nabla u_h \cdot \nabla (G - \pi G) \, dx
\]

\[
+ \int_{\Gamma} \left( \{\nabla u_h\} \cdot [G - \pi G] - \sigma [u_h] \cdot [G - \pi G] \right) \, ds + \int_{\partial \Omega} \sigma G - \pi G \, ds
\]

\[
= \int_\Omega (f + \Delta_h u_h)(G - \pi G) \, dx - \sum_{\kappa \in T_h} \int_{\partial \kappa} \nabla u_h \cdot \vec{n}(G - \pi G) \, ds
\]

\[
+ \int_{\Gamma} \left( \{\nabla u_h\} \cdot [G - \pi G] - \sigma [u_h] \cdot [G - \pi G] \right) \, ds + \int_{\partial \Omega} \sigma G - \pi G \, ds
\]

\[
= \int_\Omega (f + \Delta_h u_h)(G - \pi G) \, dx - \int_{\Gamma_{\text{int}}} \{G - \pi G\} \cdot \{\nabla u_h\} \, ds
\]

\[
- \int_{\Gamma} \sigma [u_h] \cdot [G - \pi G] \, ds + \int_{\partial \Omega} \sigma G - \pi G \, ds
\]

\[
= \int_\Omega (f + \Delta_h u_h)(G - \pi G) \, dx + \int_{\partial \Omega} \sigma (u_h - g)(G - \pi G) \, ds
\]

\[
- \int_{\Gamma_{\text{int}}} \{G - \pi G\} \cdot \{\nabla u_h\} + \sigma [u_h][G - \pi G] \, ds.
\]

Hölder’s inequality then implies

\[
I \leq \|h^2(f + \Delta_h u_h)\|_{L_\infty(\Omega)} \|h^{-2}(G - \pi G)\|_{L_1(\Omega)}
\]

\[
+ \|h\{\nabla u_h\}\|_{L_\infty(\Gamma_{\text{int}})} \|h^{-1}\{G - \pi G\}\|_{L_1(\Gamma_{\text{int}})}
\]

\[
+ (\|h\sigma [u_h]\|_{L_\infty(\Gamma_{\text{int}})} + \|h\sigma (u_h - g)\|_{L_\infty(\partial \Omega)}) \|h^{-1}[G - \pi G]\|_{L_1(\Gamma)},
\]

(5.5)
or, using standard scaled trace estimates and the shape-regularity of the mesh while recalling that $\sigma = C_\sigma r^2/h$,

$$I \lesssim \left( \|h^2(f + \Delta_h u_h)\|_{L^\infty(\Omega)} + \|h[\nabla u_h]\|_{L^\infty(\Gamma_{in})} + C_\sigma r^2\|u_h\|_{L^\infty(\Gamma_{in})} + C_\sigma r^2\|(u_h - g)\|_{L^\infty(\partial\Omega)} \right) \times \left( \|h^{-2}(G - \pi G)\|_{L^1(\Omega)} + \|h^{-1}\nabla(G - \pi G)\|_{L^1(\Omega)} \right). \tag{5.6}$$

In order to bound the term $II$, we note that $|G| = 0$ on $\Gamma$ and compute

$$II = \int_\Omega \nabla_h u_h \cdot \nabla G \, dx - \int_\Gamma \{\nabla \pi G\} \cdot [u_h] \, ds - u_h(x_0) - \int_{\partial\Omega} g \partial_{\vec{n}}(G - \pi G) \, ds. \tag{5.7}$$

Formally we would like to integrate the first term above by parts elementwise, but doing so causes difficulties for elements near $x_0$ because $G$ is singular there. We instead work as follows. Let $\kappa \in T$ be an element whose closure contains $x_0$. Let $P_\kappa$ be the set of all elements in $T$ touching $\kappa$, and similarly let $P_\kappa'$ be the set of all elements touching $P_\kappa$. Let also $\bar{T}$ be the minimal conforming refinement of $T$ defined in $\ell 3$ and let $P_\kappa$ be the region consisting of all elements in $\bar{T}$ touching $P_\kappa$. We now let $\omega$ be the function that is continuous, piecewise linear with respect to $\bar{T}$, identically 1 on $P_\kappa$, and 0 on all vertices of $\bar{\kappa}$ not lying in $P_\kappa$. $\omega$ will act as a cutoff function to separate elements that are too close to the singularity at $x_0$ and those that are far enough away. Employing the shape regularity of $T$ along with the shape regularity and conformity of $\bar{T}$, we have that $\text{dist}(x_0, \text{supp}(1 - \omega)) \gtrsim h_\kappa$. Note also that $\text{supp}(\omega) \subset P_\kappa'$.

The term $II$ measures in part the degree by which $u_h$ is not a conforming approximation to $u$, and we now accordingly employ the conforming approximation $\chi$ to $u_h$ defined in Lemma 4.2. Noting that $\omega \chi \in W^{1,\infty}(\Omega)$ and recalling that $\omega(x_0) = 1$, we have $\chi(x_0) = (\omega \chi)(x_0)$ and thus $\chi(x_0) = \int_\Omega \nabla(\omega \chi) \cdot \nabla G \, dx - \int_{\partial\Omega} \omega \chi \partial_{\vec{n}} G \, ds$. Inserting $-\chi(x_0) + \chi(x_0)$ into (5.7) and rearranging terms, we thus compute

$$II = \int_\Omega \nabla_h [(1 - \omega)u_h] \cdot \nabla G \, dx + \int_\Omega \nabla_h (\omega u_h) \cdot \nabla G \, dx - \int_\Gamma \{\nabla \pi G\} \cdot [u_h] \, ds$$

$$- \int_\Omega \nabla_h (\omega \chi) \cdot \nabla G \, dx + \int_{\partial\Omega} [\omega \chi \partial_{\vec{n}} G - g \partial_{\vec{n}}(G - \pi G)] \, ds + \chi(x_0) - u_h(x_0)$$

$$= \left[ \int_\Omega \nabla_h [(1 - \omega)u_h] \cdot \nabla G \, dx - \int_\Gamma \{\nabla \pi G\} \cdot [(1 - \omega)u_h] \, ds \right.$$

$$- \int_{\partial\Omega} [(1 - \omega)g \partial_{\vec{n}}(G - \pi G)] \, ds \right] + \left[ \int_\Omega \nabla_h [\omega(u_h - \chi)] \cdot \nabla G \, dx - \int_\Gamma \{\nabla \pi G\} \cdot [\omega u_h] \, ds + (\chi - u_h)(x_0) \right.$$

$$+ \int_{\partial\Omega} \omega[\chi \partial_{\vec{n}} G - g \partial_{\vec{n}}(G - \pi G)] \, ds \right]$$

$$=: [II_a] + [II_b]. \tag{5.8}$$

We first bound the term $II_b$. Note that $\omega(u_h - \chi)$ is a piecewise polynomial on $\bar{T}$ and that $\bar{T}$ is quasi-uniform on the support of $\omega$, so we may employ inverse estimates
Similarly, \( \omega(u_h - \chi) \) using \( h_\kappa \) as the mesh parameter (cf. (3.1)). Using Hölder’s inequality, recalling that \( 0 \leq \omega \leq 1 \), and using (4.4), we thus have

\[
\int_{\Omega} \nabla_h[\omega(u_h - \chi)] \cdot \nabla G \, dx \leq h_\kappa \|\nabla_h[\omega(u_h - \chi)]\|_{L_\infty(\tilde{P}_\kappa')} \|h^{-1}\nabla G\|_{L_1(\tilde{P}_\kappa')}
\]

\[
\lesssim \|u_h - \chi\|_{L_\infty(P_\kappa')} \|h^{-1}\nabla G\|_{L_1(\tilde{P}_\kappa')}
\]

\[
\lesssim (\|u_h\|_{L_\infty(\Gamma_{\text{int}})} + \|g - u_h\|_{L_\infty(\partial\Omega)}) \|h^{-1}\nabla G\|_{L_1(\tilde{P}_\kappa')}. \tag{5.9}
\]

We also note that

\[
\int_{\partial\Omega} \omega[\chi \partial_\vec{n} G - g \partial_\vec{n} (G - \pi G)] \, ds - \int_{\Gamma} \{\nabla \pi G\} \cdot [\omega u_h] \, ds
\]

\[
= \int_{\partial\Omega} \omega(\chi - g) \partial_\vec{n} G \, ds + \int_{\partial\Omega} \omega(g - u_h) \partial_\vec{n} (\pi G) \, ds \tag{5.10}
\]

\[
- \int_{\Gamma_{\text{int}}} \{\nabla \pi G\} \cdot [\omega u_h] \, ds.
\]

Using standard trace inequalities, recalling that \( \text{supp}(\omega) \subset P_\kappa' \) and that \( \omega|_e = 0 \) for any \( e \in \partial P_\kappa' \), and applying inverse inequalities elementwise to \( \pi G \) while recalling that \( \mathcal{T} \) and \( \tilde{T} \) are quasi-uniform on \( P_\kappa' \), we have

\[
\int_{\Gamma_{\text{int}}} \{\nabla \pi G\} \cdot [\omega u_h] \, ds \leq \sum_{e \in \Gamma_{\text{int}} \cap P_\kappa'} \|\nabla \pi G\|_{L_1(e)} \|\omega u_h\|_{L_\infty(e)}
\]

\[
\lesssim \|u_h\|_{L_\infty(\Gamma_{\text{int}} \cap P_\kappa')} \sum_{T \subset P_\kappa'} (h_T^{-1} \|\nabla \pi G\|_{L_1(T)} + \|D^2 \pi G\|_{L_1(T)}) \tag{5.11}
\]

\[
\lesssim \|u_h\|_{L_\infty(\Gamma_{\text{int}} \cap P_\kappa')} \|h^{-1}\nabla \pi G\|_{L_1(P_\kappa')}. \tag{5.12}
\]

Similarly,

\[
\int_{\partial\Omega} \omega(g - u_h) \partial_\vec{n} (\pi G) \, ds \leq \|g - u_h\|_{L_\infty(\partial\Omega)} \|h^{-1}\nabla \pi G\|_{L_1(P_\kappa')}.
\]

Recalling the representation (2.7), applying the maximum principle (2.8) to the harmonic extension of \( \omega(g - \chi) \), and applying (4.4) along with the triangle inequality yields

\[
\int_{\partial\Omega} \omega(\chi - g) \partial_\vec{n} G \, ds \leq \|\omega(g - \chi)\|_{L_\infty(\partial\Omega)} \lesssim \|g - u_h\|_{L_\infty(\partial\Omega)} + \|u_h\|_{L_\infty(\Gamma_{\text{int}})}. \tag{5.13}
\]

Collecting (5.10) through (5.13), we thus have

\[
II_h \lesssim (1 + \|h^{-1}\nabla G\|_{L_1(P_\kappa')} + \|h^{-1}\nabla \pi G\|_{L_1(P_\kappa')})
\]

\[
\times (\|u_h\|_{L_\infty(\Gamma_{\text{int}})} + \|g - u_h\|_{L_\infty(\partial\Omega)}). \tag{5.14}
\]

We now bound the term \( I_{II} \). Integrating by parts elementwise and using (5.3), we first compute

\[
\int_{\Omega} \nabla_h[(1 - \omega)u_h] \cdot \nabla G \, dx = - \int_{\Omega} (1 - \omega)u_h \Delta G + \int_{\Gamma} \{\nabla G\} \cdot [(1 - \omega)u_h] \, ds
\]

\[
+ \int_{\Gamma_{\text{int}}} \{\nabla(1 - \omega)u_h\} \cdot [\nabla G] \, ds. \tag{5.15}
\]
Recall now that $G$ is harmonic on $\text{supp}(1 - \omega)$ and that $[\nabla G] = 0$. Inserting (5.15) into (5.8) then yields

$$H_a = \int_{\Gamma} \{\nabla (G - \pi G)\} \cdot [(1 - \omega) u_h] \, ds - \int_{\partial\Omega} (1 - \omega) g \partial_n (G - \pi G) \, ds$$

$$= \int_{\Gamma_{int}} \{\nabla (G - \pi G)\} \cdot [(1 - \omega) u_h] \, ds + \int_{\partial\Omega} (1 - \omega) (u_h - g) \partial_n (G - \pi G) \, ds. \quad (5.16)$$

Applying a standard trace inequality yields

$$H_a \lesssim \left( \left\| \left[ (1 - \omega) u_h \right] \right\|_{L^\infty(\Gamma_{int})} + \left\| (1 - \omega) (u_h - g) \right\|_{L^\infty(\partial\Omega)} \right) \times \sum_{\kappa \in T : \kappa \subset \text{supp}(1 - \omega)} h_{T}^{-1} \left\| \nabla (G - \pi G) \right\|_{L^1(\kappa)} + \left\| D^2 (G - \pi G) \right\|_{L^1(\kappa)}$$

$$\lesssim \left( \left\| \left[ u_h \right] \right\|_{L^\infty(\Gamma_{int})} + \left\| u_h - g \right\|_{L^\infty(\partial\Omega)} \right) \times \left( \left\| h^{-1} \nabla (G - \pi G) \right\|_{L^1(\Omega \setminus P_\kappa)} + \left\| D^2 (G - \pi G) \right\|_{L^1(\Omega \setminus P_\kappa)} \right). \quad (5.17)$$

Collecting (5.6), (5.14), and (5.17) into (5.2) yields

$$\left| (u - u_h)(x_0) \right| \lesssim \mathcal{F}(G) \left( \left\| h^2 (f + \Delta_h u_h) \right\|_{L^\infty(\Omega)} + \left\| h [\nabla u_h] \right\|_{L^\infty(\Gamma_{int})} \right)$$

$$+ \left( 1 + C_\sigma r^2 \right) \left( \left\| u_h \right\|_{L^\infty(\Gamma_{int})} + \left\| g - u_h \right\|_{L^\infty(\partial\Omega)} \right), \quad (5.18)$$

where

$$\mathcal{F}(G) = 1 + \left\| h^{-2} (G - \pi G) \right\|_{L^1(\Omega)} + \left\| h^{-1} \nabla (G - \pi G) \right\|_{L^1(\Omega)}$$

$$+ \left\| D^2 (G - \pi G) \right\|_{L^1(\Omega \setminus P_\kappa)} + \left\| h^{-1} \nabla G \right\|_{L^1(\Omega \setminus P_\kappa)} + \left\| h^{-1} \nabla (\pi G) \right\|_{L^1(\Omega \setminus P_\kappa)}, \quad (5.19)$$

The shape regularity and local quasi-uniformity of $T$ guarantee the existence of $c_2 > c_1 > 0$ such that $P_{\kappa} \subset B_{c_2 h(x_0)}(x_0)$ and $\Omega \setminus P_\kappa \subset \Omega \setminus B_{c_1 h(x_0)}(x_0)$. Using stability and approximation properties of $\pi$ from Lemma 4.1, we thus have

$$\mathcal{F}(G) \lesssim 1 + h(x_0)^{-1} \left\| \nabla G \right\|_{L^1(B_{c_2 h(x_0)}(x_0))} + \left\| D^2 G \right\|_{L^1(\Omega \setminus B_{c_1 h(x_0)}(x_0))}.$$ \quad (5.20)

The proof of Theorem 5.1 will be complete after the following lemma is proved.

**Lemma 5.2.** Let $x_0 \in \Omega$, and let $G = G(\cdot, x_0)$. Under the assumptions of Theorem 5.1,

$$h(x_0)^{-1} \left\| \nabla G \right\|_{L^1(B_{c_2 h(x_0)}(x_0))} + \left\| D^2 G \right\|_{L^1(\Omega \setminus B_{c_1 h(x_0)}(x_0))} \lesssim 1 + \left( \ln(1/h) \right)^{\alpha_2}, \quad (5.21)$$

where $\alpha_2 = 2$ and $\alpha_3 = 1$ and the constant hidden in $\lesssim$ is independent of $x_0$ and of $h$.

**Proof.** Let $h_0 = h(x_0)$ and $B_2 = B_{c_2 h(x_0)}(x_0)$. Employing Hölder’s inequality along with (2.5) for $q = \frac{2d}{2d - 1} < \frac{d}{d - 1}$, we first compute

$$h_0^{-1} \left\| \nabla G \right\|_{L^1(B_2)} \lesssim h_0^{-1} \left( \left\| \nabla G \right\|_{L^1(B_{h_0^2(x_0)} \setminus B_2)} + \left\| \nabla G \right\|_{L^1(B_2 \setminus B_{h_0^2(x_0)})} \right)$$

$$\lesssim h_0^{-1} (h_0^2(x_0))^{1 - \frac{1}{q}} \left\| \nabla G \right\|_{L^q(\Omega)} + \left\| \nabla G \right\|_{L^1(\Omega \setminus B_{h_0^2(x_0)})}.$$ \quad (5.22)
We carry out an annular dyadic decomposition of $B_2 \setminus B_{h_0}(x_0)$. Let $d_j = 2^{j-1}h_0^2$, $j = 0, 1, \ldots$, and let $\Omega_j = \{x \in \Omega : d_j < |x-x_0| \leq d_{j+1}\}$. Thus $B_2 = \bigcup_{j=1}^{J} \Omega_j$ for some $J \approx \ln \frac{1}{h_0}$. Let also $\Omega'_j = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$. Employing H"older’s inequality, a standard Cacciopoli inequality $\|\nabla G\|_{L^2(\Omega_j)} \leq C d_j^{-1} \|G\|_{L^2(\Omega'_j)}$ for harmonic functions, and the pointwise bound (2.6) then yields

$$\|\nabla G\|_{L^2(B_2 \setminus B_{h_0}(x_0))} \leq \sum_{j=1}^{J} d_j^{d/2} \|\nabla G\|_{L^2(\Omega_j)} \leq \sum_{j=1}^{J} d_j^{d/2-1} \|G\|_{L^2(\Omega'_j)} \leq \sum_{j=0}^{J+1} d_j^{-1} \|G\|_{L^\infty(\Omega_j)} \leq \left\{ \begin{array}{ll} h_0, & d = 3, \\

h_0 \ln \frac{1}{h_0}, & d = 2. \end{array} \right.$$  \hspace{1cm} (5.23)

Thus

$$h_0^{-1} \|\nabla G\|_{L^1(B_2)} \approx \ln \frac{1}{h_0}. \hspace{1cm} (5.24)$$

In order to bound the term $\|D^2G\|_{L^1(\Omega \setminus B_{h_0}(x_0))}$ in (5.21), we employ a diadic decomposition about $x_0$. Let $B_1 = B_{c_1 h_0}(x_0)$. For $i \geq -2$ let $d_i = 2^i c_1 h_0(x_0)$, and let $\Omega_i = \{x \in \Omega : d_i < |x-x_0| < d_{i+1}\}$. Let also $J = \max\{j : \Omega_j \neq \emptyset\}$; note that $J \leq C \ln(1/h_0)$. Let also $\omega_j$ be a cutoff function which is 1 on $\Omega_j$, 0 outside of $\Omega'_j = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$, and which satisfies $\|D^k \omega_j\|_{L^\infty(\Omega)} \leq C d_j^{-k}$, $k = 0, 1, 2$. Using the fact that $G$ is harmonic on each $\Omega_j$ along with the regularity estimate (2.1), and finally using H"older’s inequality and regrouping terms, we first compute that for any fixed $1 < p < \frac{4}{3}$,

$$\|D^2G\|_{L^1(\Omega \setminus B_1)} \leq \sum_{j=0}^{J} \|D^2G\|_{L^1(\Omega_j)} \leq \sum_{j=0}^{J} d_j^{-\frac{d}{2}} \|D^2(\omega_j G)\|_{L^p(\Omega_j)} \leq C_p \sum_{j=0}^{J} d_j^{-\frac{d}{2}} \|2 \nabla G \cdot \nabla \omega_j + G \Delta \omega_j\|_{L^p(\Omega_j)} \hspace{1cm} (5.25)$$

Again employing a Cacciopoli inequality $\|\nabla G\|_{L^2(\Omega_j)} \approx d_j^{-1} \|G\|_{L^2(\Omega'_j)}$ along with (2.6), we compute for $d = 3$ that

$$\sum_{j=-1}^{J} \left( d_j^{-1} \|\nabla G\|_{L^2(\Omega_j)} + d_j^{-2} \|G\|_{L^2(\Omega_j)} \right) \approx \sum_{j=-2}^{J} d_j^{-2} \|G\|_{L^2(\Omega_j)} \leq \sum_{j=-2}^{J} d_j^{-2} d_j^{-1} \lesssim J \lesssim \ln \frac{1}{h_0}. \hspace{1cm} (5.26)$$

Similarly, for $d = 2$ we have

$$\sum_{j=-1}^{J} \left( d_j^{-1} \|\nabla G\|_{L^2(\Omega_j)} + d_j^{-2} \|G\|_{L^2(\Omega_j)} \right) \leq C \sum_{j=-2}^{J} \ln d_j \leq C \left( \frac{\ln \frac{1}{h_0}}{h_0} \right)^2. \hspace{1cm} (5.27)$$
This completes the proof of Lemma 5.2 and, thus, of Theorem 5.1.

5.2. Efficiency results. We continue by showing that the above a posteriori bound is also efficient.

Theorem 5.3. Under the same assumptions as in Theorem 5.1, the following bounds hold:

\[
\|h^2(\pi f + \Delta u_h)\|_{L^\infty(\kappa)} \lesssim \|u - u_h\|_{L^\infty(\kappa)} + \|h^2(f - \pi f)\|_{L^\infty(\kappa)},
\]

(5.28)

for all \(\kappa \in T\), with \(\pi\) denoting the orthogonal element-wise \(L^2\)-projection onto \(V_h\), and

\[
\|h|\nabla u_h|\|_{L^\infty(\kappa)} \lesssim \|u - u_h\|_{L^\infty(\kappa)} + \|h^2(f - \pi f)\|_{L^\infty(\kappa)},
\]

(5.29)

for all faces \(\kappa_1 \cap \kappa_2 = e \subset \Gamma_{\text{int}}\), for all neighbouring \(\kappa_1, \kappa_2 \in T\).

Remark 5.4. Theorem 5.3 only includes bounds for the first two terms in (5.1).

Proof. We omit some details from our proof, since the steps are mostly standard. To show (5.28), we consider the standard polynomial elemental bubble \(b_\kappa\) on \(\kappa \in T\) (see, e.g., [44]), which is continuous on \(\Omega\), such that \(b_\kappa|_\kappa \in P_s(\kappa)\) (for some positive integer \(s\), depending on the element geometry: \(s = 3\) on triangles, \(s = 4\) on tetrahedra and quadrilaterals and \(s = 6\) on hexahedra), \(b_\kappa|_{\Omega\setminus\kappa} = 0\) and \(\|b_\kappa\|_{L^\infty(\kappa)} \lesssim 1\).

Now, let \(v := (\pi f + \Delta u_h)b_\kappa\). Integration by parts yields

\[
\int_\kappa (u - u_h)\Delta v \, dx = -\int_\kappa \nabla (u - u_h) \cdot \nabla v \, dx + \int_{\partial \kappa} (u - u_h)\partial n v \, ds.
\]

(5.30)

Here we have used the fact that \(u \in L^\infty(\Gamma)\). This may be established by writing \(u = u_0 + g\), where \(u_0 \in H^1_0(\Omega)\) weakly solves \(-\Delta u = f\) and \(g\) is without loss of generality taken to be harmonic. Here \(u_0\) satisfies the conditions of Lemma 2.1 so that \(u_0 \in W^{1,q}_0\) for some \(q > d\) and so by the Sobolev embedding \(W^{1,q}_0 \hookrightarrow C^0\) we also have \(u_0 \in C^0(\Omega)\). Similarly, we have assumed that \(g \in W^{1,q}(\Omega)\) for some \(q > d\), which by Sobolev embedding implies that \(g \in C_B(\Omega) := \{v \in C(\Omega) : \max_{x \in \Omega} |v(x)| < \infty\}\) (cf. [29], p. 158). This implies that \(u \in C_B(\Omega)\), and thus \(u\) is also in \(L^\infty(\Gamma)\) since \(u = g\) is bounded on \(\partial \Omega\).

Using the fact that \(B(u, v) = \int_\kappa fv \, dx\), for \(v\) as above, along with another integration by parts, we then arrive at

\[
\int_\kappa (u - u_h)\Delta v \, dx = \int_\kappa (f + \Delta u_h)v \, dx + \int_{\partial \kappa} (u - u_h)\partial n v \, ds,
\]

(5.31)

recalling that \(v = 0\) on \(\partial \kappa\). We can then immediately deduce

\[
\|(\pi f + \Delta u_h)\sqrt{b_\kappa}\|_{L^2(\kappa)}^2 = \int_\kappa (u - u_h)\Delta v \, dx - \int_\kappa (f - \pi f)v \, dx - \int_{\partial \kappa} (u - u_h)\partial n v \, ds.
\]

(5.32)

From the equivalence of norms in finite dimensional normed spaces and the structure of \(b_\kappa\), (see, e.g., [44] for details) and a standard inverse estimate we have

\[
\|h^2(\pi f + \Delta u_h)\|_{L^\infty(\kappa)} \lesssim h_\kappa^{4-d}\|\pi f + \Delta u_h\|_{L^2(\kappa)}^2 \lesssim h_\kappa^{4-d}\|(\pi f + \Delta u_h)\sqrt{b_\kappa}\|_{L^2(\kappa)}^2,
\]

(5.33)
respectively. Combining (5.33) with (5.32), using standard inverse estimates and recalling the shape-regularity assumption, we deduce
\[ \|h^2(\pi f + \Delta u_h)\|_{L_\infty(\kappa)} \lesssim \|u - u_h\|_{L_\infty(\kappa)} + \|h^2(f - \pi f)\|_{L_\infty(\kappa)}, \] (5.34)
which is (5.28).

To show (5.29), let \( \tilde{k}_1, \tilde{k}_2 \in \tilde{T} \), such that \( e = \tilde{k}_1 \cap \tilde{k}_2 \), where, as above, \( \tilde{T} \) is the minimal conforming refinement of \( T \). Note that then, \( \tilde{k}_i \subset \kappa_i, i = 1, 2 \). Let \( b_e \) denote the usual polynomial edge bubble (see, e.g., [44] for more details), which is continuous on \( \Omega \), such that \( b_e\|_{\Omega \setminus (\tilde{k}_1 \cup \tilde{k}_2)} = 0 \) and \( \|b_e\|_{L_\infty(\tilde{k}_1 \cup \tilde{k}_2)} \leq 1 \). Let also \( \phi_e : \tilde{k}_1 \cup \tilde{k}_2 \to \mathbb{R} \) be such that \( \phi_e|_e = [\nabla u_h]_e \) and \( \nabla \phi_e \cdot \vec{n} = 0 \) on \( \tilde{k}_1 \cup \tilde{k}_2 \) (that is \( \phi_e \) is constant along the direction of the normal vector \( \vec{n} \) to \( e \)).

To simplify the notation, let \( w := \phi_e b_e \) and \( \tilde{k}_{12} := \tilde{k}_1 \cup \tilde{k}_2 \). Integration by parts yields
\[ \int_{\tilde{k}_{12}} (u - u_h)\Delta w \, dx = -\sum_{i=1,2} \left( \int_{\tilde{k}_i} \nabla (u - u_h) \cdot \nabla w \, dx + \int_{\partial \tilde{k}_i} (u - u_h)\partial_n w \, ds \right). \] (5.35)
Observing that \( B(u, w) = \int_{\tilde{k}_{12}} fw \, dx \), another integration by parts leads to
\[ \int_{\tilde{k}_{12}} (u - u_h)\Delta w \, dx = \sum_{i=1,2} \left( \int_{\tilde{k}_i} (f + \Delta u_h)w \, dx \
+ \int_{\partial \tilde{k}_i} (u - u_h)\partial_n w \, ds \right) - \int_{e} [\nabla u_h]w \, ds . \] (5.36)
Here we have also used that \( w = 0 \) on \( \partial(\tilde{k}_1 \cup \tilde{k}_2) \setminus e \).

The equivalence of norms in finite dimensional normed spaces and the structure of \( b_e \), (see, e.g., [44]), along with a standard inverse estimate imply
\[ \|h[\nabla u_h]\|_{L_\infty(e)}^2 \lesssim h_e^{3-d} \|[\nabla u_h]\|_{L_2(e)}^2 \lesssim h_e^{3-d} \int_e [\nabla u_h]w \, ds. \] (5.37)
Combining (5.37) with (5.36), using standard inverse estimates and recalling the shape-regularity assumption, we then deduce
\[ \|h[\nabla u_h]\|_{L_\infty(\tilde{k}_{12})} \lesssim \|h^2(f + \Delta u_h)\|_{L_\infty(\tilde{k}_{12})} + \|u - u_h\|_{L_\infty(\tilde{k}_{12})} . \] (5.38)
The bound (5.29) follows by applying (5.28). □

6. Estimates for conforming finite element methods. We now describe how our techniques above can be applied to obtain improved estimates for continuous Galerkin methods on polyhedral domains. For simplicity of presentation we now assume that \( T \) is a conforming, shape-regular simplicial decomposition of a polyhedral domain \( \Omega \) in \( \mathbb{R}^d, d = 2, 3 \). Let also \( S_h \) be a space of continuous piecewise polynomials of degree \( r \). Finally, let \( u_h \in S_h \) have prescribed boundary conditions (also denoted by \( u_h \)) and satisfy
\[ \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx = \int_{\Omega} fv_h \, dx, \quad v_h \in S_h \cap H_0^1(\Omega). \] (6.1)

Theorem 6.1. Assume that \( \Omega \subset \mathbb{R}^d, d = 2, 3 \), is a polyhedral domain and that \( u_h \) is a conforming finite element approximation to a solution \( u \) of (1.1). Then
\[ \|u - u_h\|_{L_\infty(\Omega)} \leq C_{\ell_h,d}(\|h^2(f + \Delta u_h)\|_{L_\infty(\Omega)} + \|h[\nabla u_h]\|_{L_\infty(\Gamma_{in})}) \] (6.2)
Here $\ell_{h, d} = (\ln(1/h))^\alpha d$, where $\alpha_2 = 2$ and $\alpha_3 = 1$.

Proof. (Sketch of Proof) Setting $e = u - u_h$, we combine (2.4) and (2.7) to find that for $x_0 \in \Omega$,

$$e(x_0) = \int_\Omega \nabla e \cdot \nabla G(x_0, y)dx - \int_{\partial\Omega} \partial_\Gamma G(x_0, y)(g(y) - u_h(y))ds. \quad (6.3)$$

Applying the maximum principle (2.8) to the harmonic extension of $g - u_h$ yields

$$\left| - \int_{\partial\Omega} \partial_\Gamma G(x_0, y)(g(y) - u_h(y))ds \right| \leq \|g - u_h\|_{L^\infty(\partial\Omega)}. \quad (6.4)$$

In order to bound the first term in (6.3), we use Galerkin orthogonality to insert the conforming Scott-Zhang interpolant $I_hG$ of $G$, integrate by parts elementwise, use the Cauchy-Schwarz inequality while recognizing that $G - I_hG = 0$ on $\partial\Omega$, and then use a scaled trace inequality to find that

$$\int_\Omega \nabla e \cdot \nabla G(x_0, y)dx = \sum_{T} \int_T (f + \Delta u_h)(y)(G - I_hG)(x_0, y)dx$$

- $\frac{1}{2} \int_{\partial\Omega} [\nabla u_h] \cdot \nabla G(x_0, y)ds$

$\leq \sum_{T} \left( \|f + \Delta u_h\|_{L^\infty(\partial\Omega)} \|G - I_hG\|_{L^1(\partial\Omega)} + \|\nabla u_h\|_{L^\infty(\partial\Omega)} \|G - I_hG\|_{L^1(\partial\Omega)} \right)$

$\leq \left( \|f + \Delta u_h\|_{L^\infty(\partial\Omega)} + \|\nabla u_h\|_{L^\infty(\partial\Omega)} \|G - I_hG\|_{L^1(\partial\Omega)} \right)$

$\times \sum_{T} \left( h_{h, T}^{-2}\|G - I_hG\|_{L^1(\partial\Omega)} + h_{h, T}^{-1}\|\nabla(G - I_hG)\|_{L^1(\partial\Omega)} \right). \quad (6.5)$

Employing approximation properties and the finite overlap of element patches in a standard way, we see that $\sum_{T} \left( h_{h, T}^{-2}\|G - I_hG\|_{L^1(\partial\Omega)} + h_{h, T}^{-1}\|\nabla(G - I_hG)\|_{L^1(\partial\Omega)} \right)$ is bounded by the left-hand-side of (5.21) in Lemma 5.2. Employing Lemma 5.2 thus implies the desired bound. □

We make some brief comments concerning (6.2). As noted in the introduction, Theorem 6.1 generalizes previous a posteriori maximum-norm estimates in the literature by giving fully a posteriori results that admit arbitrary polyhedral domains (not just Lipschitz domains), and also improve the exponent in the logarithmic factor from 4/3 to 1 in the case $d = 3$.

It also should be noted that the contribution of the boundary term $\|g - u_h\|_{L^\infty(\partial\Omega)}$ is bounded more sharply in (6.2) than in the corresponding bound (5.1) for discontinuous Galerkin methods. In particular, $\|g - u_h\|_{L^\infty(\partial\Omega)}$ is multiplied by a logarithmic factor and an unknown constant in the discontinuous Galerkin case, whereas in (6.2) this term appears with a constant of 1, as in [39]. This difference is due at least in part to the fact that the boundary conditions are imposed strongly in the standard Galerkin method but weakly in the dG method.

7. Numerical Examples. In this section we describe computational tests using our estimators on two- and three-dimensional domains. The finite element package deal.ii [5] with suitable modifications in order to accommodate $L^\infty$ error estimators was used for all computations. As is standard in deal.ii, box (quadrilateral and hexahedral) elements were used. A standard adaptive feedback loop was employed with a “maximum strategy” used for element marking.
7.1. **Two-dimensional test.** In this section we describe standard computational tests on a two-dimensional L-shaped domain. The known test solution \( u \), right-hand-side \( f \), and boundary data \( g \) were chosen in order to ensure that \( u \) has the correct singular asymptotics of \( r^{2/3} \) near the reentrant corner. Figure 7.1 displays an adapted mesh using cubic elements, while Figure 7.2 displays optimal-order decrease of errors and estimators along with effectivity indices for linear, quadratic, and cubic elements. Note that the effectivity indices (which measure the ratio of the estimator to the error) are quite stable for fixed polynomial degree \( r \) as the mesh is refined, but increase substantially as the polynomial degree \( r \) increases. This is at least partially due to the fact that our error estimator in Theorem 5.1 includes scaling by \( r^2 \) due to inclusion of the penalty constant, but we do not specify scaling with \( r \) of the constant hidden in "\( \lesssim \)". As indicated by our numerical experiments, development of a "\( p \)"-version of our error estimator would be an interesting further investigation.

![Fig. 7.1. Mesh on the L-shaped domain after 20 adaptive iterations using cubic elements.](http://example.com/mesh.png)

![Fig. 7.2. Error decrease (left) and effectivity indices (right) on the L-shaped domain.](http://example.com/error.png)

7.2. **Three-dimensional tests.** We now describe a test problem on a three-dimensional non-Lipschitz domain. A “two-brick” domain \( \Omega \) is pictured in Figure 7.3 (cf. [36]; the example is also attributed to Maz’ya). The domain is not Lipschitz at the origin \( O \). However, solutions to elliptic problems on \( \Omega \) may still be analyzed using standard techniques for boundary value problems on polyhedral domains. We briefly describe some solution features; we refer to slides of Monique Dauge found at [http://example.com/duage.slides](http://example.com/duage.slides).
Assuming for simplicity that \( g = 0 \), the solution \( u \) to (1.1) may be decomposed into regular and singular parts as \( u = u_{\text{reg}} + \Sigma_{v \in \mathcal{V}} u_v + \Sigma_{e \in \mathcal{E}} u_e \). Here \( \mathcal{V} \) is the set of vertices on \( \partial \Omega \) and \( \mathcal{E} \) is the set of edges. Letting \( \rho_v \) be the (positive) distance to \( v \in \mathcal{V} \), we have \( u_v = \rho_v \lambda v \psi_v(\theta) \). Here \( \theta \) is the spherical angle. Also, \( \lambda \) is the positive root of \( \lambda^2 + \lambda = \mu_v \) and \( (\mu_v, \psi_v) \) is the eigenvalue-eigenvector pair of the Dirichlet Laplace-Beltrami operator on a spherical “cap” about \( v \). The spherical cap may be formed by intersecting a sphere centered at \( v \) with \( \Omega \) and then scaling to unit size. See Figure 7.4 for the spherical cap corresponding to the vertex \( O \) in Figure 7.3.

A similar description applies to edges, but here a precise formula may be given. If the interior opening angle at an edge \( e \) is \( \omega \), let \( \lambda_e = \frac{\pi}{\omega} \) and let \( r \) be the distance to \( e \). Then \( u_e = r^2 \sin(\lambda \theta) \), where \( \theta \) is the polar angle about \( e \). At the edges \( e_1 \) and \( e_2 \) pictured in Figure 7.3, we thus have \( \omega = 3\pi/2 \) and \( u_e = r^{2/3} \sin(2\theta/3) \).
Values of $\mu_v$ are known analytically for certain simple configurations (e.g., for wedge-shaped openings), but not to our knowledge for the non-Lipschitz vertex $O$ of the two-brick domain. We thus performed an adaptive eigenvalue computation in order to assess the strength of the natural singularity at $O$. Using techniques for piecewise linear adaptive surface FEM introduced in [17] and simple residual-based a posteriori estimators for eigenvalue problems (cf. [32]), we adaptively solved the eigenvalue problem $-\Delta_\Gamma \psi = \lambda \psi$ on $\Gamma$ for the smallest eigenvalue. We employed a version of the software package iFEM ([12]) which was modified to accommodate eigenvalue computations.

Our computation yielded $\mu_O \approx 3.907272087424444$ on a mesh of roughly $4 \times 10^5$ degrees of freedom and with an estimated error of about $4.944490983067972 \times 10^{-5}$. Computational tests on spherical caps with known eigenvalues confirm that our estimator overestimates actual errors by a small factor (roughly 4). Because Dirichlet eigenvalues decrease monotonically to the actual value as the mesh is refined, we can conclude that $3.907222638324613 \leq \mu_O \leq 3.907272087424444$. Correspondingly, solving $\lambda^2 + \lambda = \mu_O$ yields $1.538926 \leq \lambda_O \leq 1.538939$. Letting $\rho$ be the positive distance to $O$, we have $u_O \sim \rho^{\lambda_O} \psi_O$ near $O$. While our eigenvalue code also computes an approximation to $\psi_O$ (which is color-coded in Figure 7.4), it is difficult to transfer this information to the solver we used on $\Omega$. However, it is easy to verify certain singular asymptotics computationally. In particular, near the nonconvex corners at the north pole and rear of the cap we found $\psi_O \sim s^{2/3}$, where $s$ is the (geodesic) distance to the relevant corner. These asymptotics correspond precisely to the expected singularities for opening angles of $3\pi/2$.

By way of comparison, the eigenvalue for the Dirichlet Laplace-Beltrami problem for the spherical cap at $V_1$ is known, and leads to a singularity of the form $\rho_{V_1}^{5/3}$ with $\rho_{V_1}$ the distance to $V_1$. The other vertices of $\Omega$ lead to less singular terms by monotonicity of Dirichlet eigenvalues, so $O$ is the most singular vertex. We note however that according to the formula $u \in H^s(\Omega)$ with $s < \min(7/2, \min_{v \in V} \lambda_v + 3/2, \min_{e \in E} \lambda_e + 1)$, the edge singularities place more severe restrictions on solution regularity, since $\min_{e \in E} \lambda_e + 1 = 5/3$ while $\min_{v \in V} \lambda_v +\frac{3}{2} = \lambda_O + 3/2 > 3$. To emphasize this point, it is thus the edge singularities which place the greatest limits on solution regularity and adaptive convergence rates, not the non-Lipschitz nature of the domain.

We performed numerical tests involving an unknown exact solution $u$, and with $g = 0$ and $f = 1$. As can be seen in Figure 7.5, the strongest refinement occurred along the edges $e_1$ and $e_2$, as is expected based on the discussion above. Moderate secondary grading can be seen at the other singular points (vertices and edges). Note also that we expect that the AFEM will converge at rate $s$ if $L_\infty$ compactly imbeds into a Besov space containing $u$ and having smoothness index $ns$. Taking $n = 3$, this roughly speaking translates to $L_p$ integrability of $D^{3s}u$ for $3s - 3/p > 0$, or $s > 1/p$. Solving this relationship with the edge singularities at $e_1$ and $e_2$ yields $s < 2/3$. $L_\infty$ convergence with $s = 2/3$ is generally already possible for linear elements, so using higher polynomial degree offers little advantage in this problem. This is confirmed in Figure 7.6 where the estimators for quadratic and cubic elements are seen to decrease with rate 2/3 only.

REFERENCES

Fig. 7.5. Computational mesh for the two-brick domain. The analytical solution is unknown.

Fig. 7.6. Error reduction on the two-brick domain with unknown analytical solution.


