1.3-Vector-Valued Functions and Parametrized Curves

Definitions:

Recall: Function- A rule that relates a set of inputs \( x \) to a set of outputs \( y \) such that there is at most 1 output for each input.
\[ f : \mathbb{R} \to \mathbb{R} \]
Domain Range

Vector-Valued Function- a rule that relates a set of real inputs \( t \) to a set of vector outputs \( \vec{r}(t) \) such that there is at most 1 output for each input.
\[ \vec{r} : \mathbb{R} \to \mathbb{R}^2 \]

\[ \vec{r}(t) = t\hat{e} + t^2\hat{j} \]
\[ \vec{r}(3) = 3\hat{e} + 9\hat{j} \]

Parametrized Curve-

A graph of a vector function (points corresponding to output only)

\[ \vec{r}(t) = t\hat{e} + t^2\hat{j} \]
\[ x = t \]
\[ y = t^2 \]

\[ \vec{r}(3) = 3\hat{e} + 9\hat{j} \], so we plot the point \((3, 9)\)

\[ t = \text{parameter (when)} \]
\[ (x, y) = \text{point (where)} \]
Vector/Parametric Equations of Lines

\[ \vec{r}_0 : \text{a vector corresponding to a point on line} \]
\[ \vec{v} : \text{a "direction vector" (vector// to line)} \]

\[ \vec{r}(t) = \vec{r}_0 + t \vec{v} \]

*Vector Equation of Line

**Parametric Equations**

\[ x = \_ \quad (\text{t component}) \]
\[ y = \_ \quad (\text{j component}) \]

Other Parametrized Curves

graphs of \( y = f(x) \)

\[ x = t \quad y = f(t) \]

"round things" (circles/ellipses)

Unit Circle:

\[ x = \cos t \quad y = \sin t \]

(parametrization involving trig functions: use identities to "eliminate the parameter")
Examples:

Find the Cartesian equation of the curve parametrized by \( x = \sqrt{t}, \ y = 2t + 4 \) and sketch the graph.

(\textit{eliminate the parameter})

\[
x > 0 \quad x = \sqrt{t} \quad \text{Solve for } t: \quad t = x^2
\]

\[
y = 2t + 4
\]

Correct: \( y = 2x^2 + 4; \ x > 0 \)
Describe the motion of a particle with position \( r(t) = \langle 2 \cos t, -5 \sin t \rangle \), \( 0 \leq t \leq 2\pi \).

1. **Start:**
   \[ r(0) = (2 \cos 0)\hat{i} - (5 \sin 0)\hat{j} = 2\hat{i} \]

2. **End:**
   \[ r(2\pi) = (2 \cos 2\pi)\hat{i} - (5 \sin 2\pi)\hat{j} = 2\hat{i} \]

3. **Path (Ellipse parameter):**
   \[ \sin^2 t + \cos^2 t = 1 \]
   \[ \frac{x^2}{25} + \frac{y^2}{4} = 1 \]
   \[ x = 2 \cos t \quad y = -5 \sin t \]
   \[ \frac{x}{5} = \cos t \quad \frac{y}{4} = \sin t \]
   \[ \frac{x^2}{25} = \cos^2 t \quad \frac{y^2}{4} = \sin^2 t \]

4. **Direction:**
   Let \( t = \frac{\pi}{2} \)
   \[ r \left( \frac{\pi}{2} \right) = (2 \cos \frac{\pi}{2})\hat{i} - (5 \sin \frac{\pi}{2})\hat{j} = -5\hat{j} \]
   Clockwise
Find vector and parametric equations of the line passing through the points \((-4, 2)\) and \((2, 14)\)

Point: \( \vec{r}_0 = -4\hat{i} + 2\hat{j} \)

Direction vector: \( \vec{v} = (2\hat{i} + 14\hat{j}) - (-4\hat{i} + 2\hat{j}) \)
\[ \vec{v} = 6\hat{i} + 12\hat{j} \]

\[ \vec{r}(t) = \vec{r}_0 + t\vec{v} \]

\[ \vec{r}(t) = (-4\hat{i} + 2\hat{j}) + t(6\hat{i}+12\hat{j}) \]

\[ \vec{r}(t) = (-4 + 6t)\hat{i} + (2+12t)\hat{j} \]

Vector Equation

Vector Equation

Parametric Equation

\( x = -4 + 6t \)
\( y = 2 + 12t \)
A water balloon is thrown with an initial velocity of 15 meters per second at an angle of elevation of 30°. Soon you will be able to derive the following parametric equations for the motion of the balloon:

\[
x = \frac{15\sqrt{3}}{2} t, \quad y = \frac{15}{2} t - 4.9t^2
\]

Determine how far away the balloon will strike the ground and find the Cartesian equation of the balloon’s motion.

a) When \( y = 0 \) what is \( x \)?

\[
\frac{15}{2} t - 4.9t^2 = 0 \quad \text{solve for } t
\]

\[
\frac{15}{2} t (15 - 9.8t) = 0
\]

\[
t \neq 0 \quad t = \frac{15}{9.8} \approx 1.53 \text{ seconds until hits ground}
\]

\[
x = \frac{15\sqrt{3}}{2} (1.53) \approx 19.9 \text{ meters}
\]

b) Eliminate the parameter

\[
x = \frac{15\sqrt{3}}{2} t \quad \text{solve for } t
\]

\[
t = \frac{2x}{15\sqrt{3}} \quad \text{sub into } y
\]

\[
y = \frac{15}{2} \left( \frac{2x}{15\sqrt{3}} \right) - 4.9 \left( \frac{2x}{15\sqrt{3}} \right)^2
\]

Parabola