1.3-Vector-Valued Functions and Parametrized Curves

Definitions:

Recall: Function- A rule that relates a set of inputs (x) to a set of outputs (y) such that there is at most 1 output for each input.

$$f : \underset{\text{Domain}}{\mathbb{R}} \to \underset{\text{Range}}{\mathbb{R}}$$

Vector-Valued Function- a rule that relates a set of real inputs (t) to a set of vector outputs $(\vec{r}(t))$ such that there is at most 1 output for each input.

$$\vec{r} : \mathbb{R} \to \mathbb{R}^2$$

Ex $\vec{r}(t) = t\vec{i} + t^2\vec{j}$

$\vec{r}(3) = 3\vec{i} + 9\vec{j}$

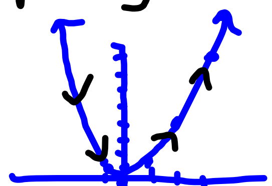
Parametrized Curve-
A graph of a vector function (points corresponding to outputs only)

$\vec{r}(t) = t\vec{i} + t^2\vec{j}$

$x = t$

$y = t^2$

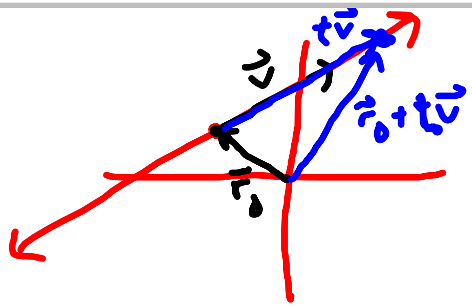
$\vec{r}(3) = 3\vec{i} + 9\vec{j}$, so we plot the point $(3, 9)$



t = parameter (when)

(x, y) = point (where)

Vector/Parametric Equations of Lines



$\vec{r}_0$: a vector corresponding to a point on line

$\vec{v}$: a "direction vector" (vector // to line)

* $\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}}$ * Vector Equation of Line

Parametric Equations

$x =$ _____ ($\vec{i}$ component)

$y =$ _____ ($\vec{j}$ component)

Other Parametrized Curves

graphs of $y=f(x)$ $\quad x=t$, $y=f(t)$

"round things" (circles/ellipses)

Unit Circle: $x = 1\cos t$, $y = 1\sin t$

(parametrization involving trig functions: use identities to "eliminate the parameter")

Examples:

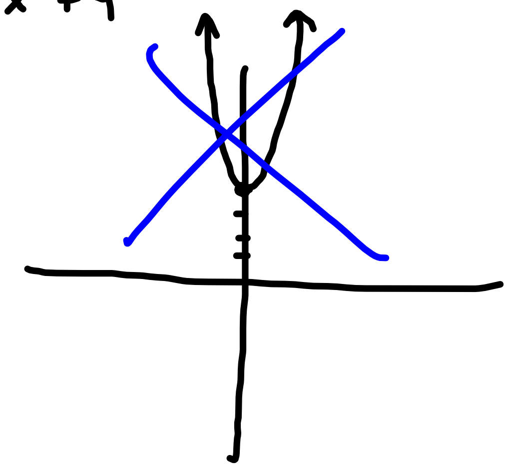
Find the Cartesian equation of the curve parametrized by $x = \sqrt{t}$, $y = 2t + 4$ and sketch the graph. (eliminate the parameter)

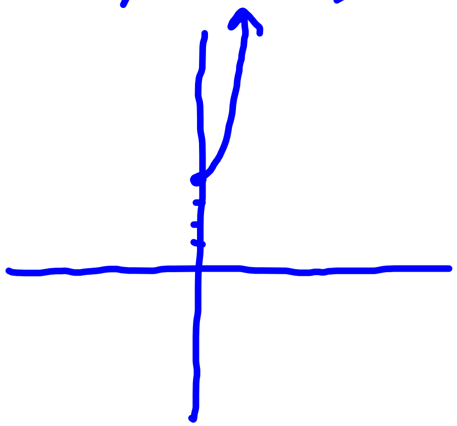
$x \geq 0$ * $x = \sqrt{t} \longrightarrow$ Solve for t: $t = x^2$

$y = 2t + 4$

$y = 2x^2 + 4$

Correct: $y = 2x^2 + 4$; $x \geq 0$

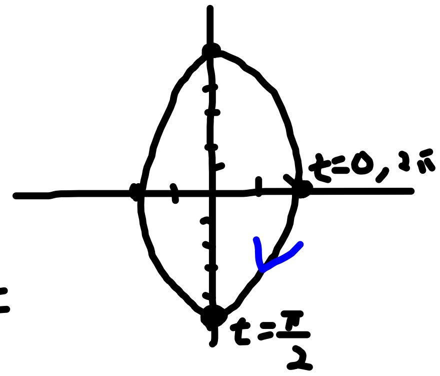




$(2\cos t)\vec{i} - (5\sin t)\vec{j}$

Describe the motion of a particle with position $\mathbf{r}(t) =< 2\cos t, -5\sin t >$, $\underline{0} \le t \le 2\pi$.

Start



① Start :

t=0: $\vec{r}(0) = (2\cos 0)\vec{i} - (5\sin 0)\vec{j} = 2\vec{i}$ (2,0)

② End:
t=2π

$\vec{r}(2\pi) = (2\cos 2\pi)\vec{i} - (5\sin 2\pi)\vec{j} = 2\vec{i}$ (2,0)

③ Path (Elim parameter)

$\sin^2 t + \cos^2 t = 1$

$x = 2\cos t$ $\quad$ $y = -5\sin t$

$\frac{x}{2} = \cos t$ $\quad$ $\frac{y}{-5} = \sin t$

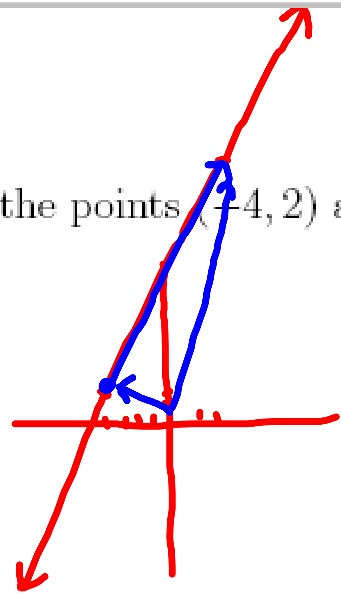
$\frac{x^2}{4} = \cos^2 t$ $\quad$ $\frac{y^2}{25} = \sin^2 t$

Ellipse $\frac{y^2}{25} + \frac{x^2}{4} = 1$

④ Direction : Let $t = \frac{\pi}{2}$

$\vec{r}\left(\frac{\pi}{2}\right) = \left(2\cos\frac{\pi}{2}\right)\vec{i} - \left(5\sin\frac{\pi}{2}\right)\vec{j} = -5\vec{j}$ (0,-5)

Clockwise



Find vector and parametric equations of the line passing through the points $(-4, 2)$ and $(2, 14)$

point: $\vec{r}_0 = -4\vec{i} + 2\vec{j}$

direction vector: $\vec{v} = (2\vec{i} + 14\vec{j}) - (-4\vec{i} + 2\vec{j})$

$$\vec{v} = 6\vec{i} + 12\vec{j}$$

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

$$\boxed{\vec{r}(t) = (-4\vec{i} + 2\vec{j}) + t(6\vec{i} + 12\vec{j})}$$ Vector Equation

$$(-4\vec{i} + 2\vec{j}) + (6t\vec{i} + 12t\vec{j})$$

$$\vec{r}(t) = (-4 + 6t)\vec{i} + (2 + 12t)\vec{j}$$ Parametric Equation

$$x = -4 + 6t$$
$$y = 2 + 12t$$

point $\quad \vec{v}$

A water balloon is thrown with an initial velocity of 15 meters per second at an angle of elevation of 30°. Soon you will be able to derive the following parametric equations for the motion of the balloon:

$$x = \frac{15\sqrt{3}}{2}t,\ y = \frac{15}{2}t - 4.9t^2$$

Determine how far away the balloon will strike the ground and find the Cartesian equation of the balloon's motion.

a) when $y=0$ what is x?

$\frac{15}{2}t - 4.9t^2 = 0$ solve for t

$\frac{1}{2}t(15 - 9.8t) = 0$

$t=0 \quad t = \frac{15}{9.8} \approx 1.53$ seconds until hits ground

$x = \frac{15\sqrt{3}}{2}(1.53) \approx \boxed{19.9 \text{ meters}}$

b) eliminate the parameter

$x = \frac{15\sqrt{3}}{2}t$ solve for t

$t = \frac{2x}{15\sqrt{3}}$ subs into y

$$\boxed{y = \frac{15}{2}\left(\frac{2x}{15\sqrt{3}}\right) - 4.9\left(\frac{2x}{15\sqrt{3}}\right)^2}$$

Parabola