2.3-Analytic Computation of Limits

Properties of Limits See pp 91-93. Basis for the techniques used in the following examples.

Use the properties to compute $\lim\limits_{x\to5} x^2 \cancel{+} - 4x + 3$.

$$= \lim_{x\to5} x^2 - \lim_{x\to5} 4x + \lim_{x\to5} 3$$

$$= \lim_{x\to5} x^2 - 4\lim_{x\to5} x + \lim_{x\to5} 3$$

$$= \left(\lim_{x\to5} x\right)^2 - 4\lim_{x\to5} x + \lim_{x\to5} 3$$

$$= 5^2 - 4(5) + 3 \quad *$$

Subs $x=5$ into function

$$= \boxed{8}$$

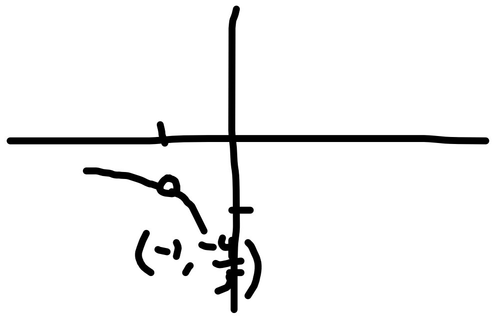
Compute $\lim_{x \to -1} \frac{x^2+6x+5}{x^2-3x-4}$

$$= \cancel{\frac{(-1)^2+6(-1)+5}{(-1)^2-3(-1)-4}} = \frac{0}{0}$$

Simplify the function

$$= \lim_{x \to -1} \frac{(x+5)\cancel{(x+1)}}{\cancel{(x+1)}(x-4)}$$

$$= \frac{-1+5}{-1-4} = \frac{4}{-5} = \boxed{-\frac{4}{5}}$$



Square Root

Compute $\lim_{y\to 9} \frac{\sqrt{y}-3}{y-9} = \frac{\sqrt{9}-3}{9-9}$ $\frac{0}{0}$

$$= \lim_{y\to 9} \frac{(\sqrt{y}-3)(\sqrt{y}+3)}{(y-9)(\sqrt{y}+3)}$$ Conjugate

$$= \lim_{y\to 9} \frac{\cancel{y-9}}{\cancel{(y-9)}(\sqrt{y}+3)} = \frac{1}{\sqrt{9}+3} = \boxed{\frac{1}{6}}$$

Compute $\lim_{x \to 3^-} \frac{1}{x-3} = \frac{1}{0}$

$\frac{\text{nonzero}}{0} = \begin{array}{l} +\infty \\ -\infty \\ \text{DNE (2 sided only)} \end{array}$

Numerical View:

$x = 2.9 \quad \frac{1}{2.9-3} = -10$

$x = 2.99 \quad \frac{1}{2.99-3} = -100$

$x = 2.999 \quad \frac{1}{2.999-3} = -1000$

$$\lim_{x \to 3^-} \frac{1}{x-3} = -\infty$$

Vertical Asymptote

($y \to \pm\infty$ as $x \to a^{\text{right or left}}$)

Limits of Vector Functions:

if $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ then

$$\lim_{t\to a} \vec{r}(t) = \left(\lim_{t\to a} x(t)\right)\vec{i} + \left(\lim_{t\to a} y(t)\right)\vec{j}$$

Compute $\lim\limits_{t\to 1} \mathbf{r}(t)$, where $\mathbf{r}(t) = \left(\dfrac{t^2+2t}{t+1}\right)\mathbf{i} + \left(\dfrac{t^4-1}{t-1}\right)\mathbf{j}$.

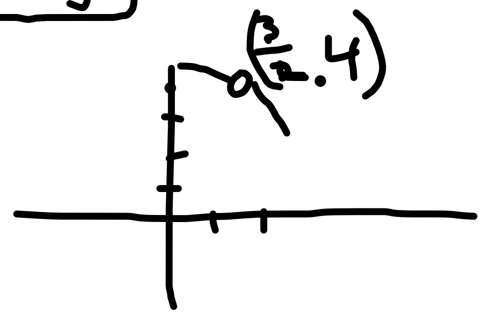
$$= \left(\lim_{t\to 1} \frac{t^2+2t}{t+1}\right)\vec{i} + \left(\lim_{t\to 1} \frac{t^4-1}{t-1}\right)^{\frac{0}{0}}\vec{j}$$

$$= \frac{1^2+2\cdot 1}{1+1}\vec{i} + \left(\lim_{t\to 1} \frac{(t^2+1)(t^2-1)}{t-1}\right)\vec{j}$$

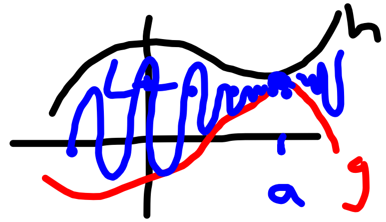
$$= \frac{3}{2}\vec{i} + \left(\lim_{t\to 1} \frac{(t^2+1)(t+1)\cancel{(t-1)}}{\cancel{t-1}}\right)\vec{j}$$

$$= \boxed{\frac{3}{2}\vec{i} + 4\vec{j}}$$

Graphically:



Squeeze Theorem: If $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$



Example: Compute $\lim_{x\to 0} x^4 \cos\left(\frac{1}{x}\right)$

$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$ Multiply by x^4

$-x^4 \leq x^4\cos\left(\frac{1}{x}\right) \leq x^4$ ($x^4 \geq 0$ no ineq changes)

$\lim_{x\to 0} -x^4 = 0$

$\lim_{x\to 0} x^4 = 0$

$\therefore$ by Squeeze Thm $\lim_{x\to 0} x^4\cos\left(\frac{1}{x}\right) = 0$