2.5-Continuity

Definitions: A function $f$ is continuous at $x = a$ if and only if $\lim_{{x \to a}} f(x) = f(a)$

$f$ is continuous from the left at $x = a$ if and only if $\lim_{{x \to a^-}} f(x) = f(a)$

$f$ is continuous from the right at $x = a$ if and only if $\lim_{{x \to a^+}} f(x) = f(a)$

$f$ is continuous on the interval $[a, b]$ if and only if $f$ cts on $(a, b)$, $f$ cts from the right at $x = a$, $f$ cts from the left at $x = b$.

Definition: $f$ has a removable discontinuity at $x = a$ if and only if there is a function $g$ such that $g$ is cts at $x = a$ and $g(x) = f(x)$ for all $x \neq a$.

Theorems:

If $\lim_{{x \to a}} g(x) = b$ and $f$ is continuous at $b$, then $\lim_{{x \to a}} f(g(x)) = f(b)$

Intermediate Value Theorem-

If $f$ cts on $[a, b]$ and $N$ is between $f(a)$ and $f(b)$, then there is a $c$ between $a$ and $b$ such that $f(c) = N$ (i.e. there is a solution to $f(x) = N$)
Then: polynomials (and most "ordinary" functions) are cts on its domain.

Examples:

Find the values of $x$ for which $f(x) = \frac{x^2 - 9}{x^2 - 5x + 6}$ is not continuous. Determine which, if any, of these discontinuities are removable.

- $f(x)$ is not continuous for all $x$ except when $x^2 - 5x + 6 = 0$.
- $(x-2)(x-3) = 0$.
- $x = 2, x = 3$.

For $x = 2$,

$$\lim_{x \to 2} \frac{x^2 - 9}{x^2 - 5x + 6} = \frac{5}{0} \quad \text{Vert Asym}$$

Not removable

For $x = 3$,

$$\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 5x + 6} = \frac{0}{0}$$

Removeable Discontinuity $g(3) = 6$
Determine whether the function

\[ f(x) = \begin{cases} 
2x - 3 & \text{if } x \leq 2 \\
x^2 & \text{if } x > 2 
\end{cases} \]

is continuous at \( x = 2 \) or not and why. Sketch the graph of the function.

3-point checklist:

1. Is \( f(2) \) defined? \( f(2) = 2(2) - 3 = 1 \)
2. Does \( \lim_{x \to 2^-} f(x) \) exist?
3. \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2x - 3 = 2(2) - 3 = 1 \)
   (\( f \) cts from left)
4. \( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 2^2 = 4 \)
5. Are they equal (limit = y-value)?

\( f \) is not cts at \( x = 2 \) since limit does not exist (left ≠ right)
Compute \( \lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 1} \)

\[
= \left| \lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 1} \right| \\
= \left| \lim_{x \to 1} \frac{(x+3)(x-1)}{(x+1)(x-1)} \right| \\
= \left| \frac{4}{2} \right| = 2
\]

\( f(x) = |x| \)

abs val function is cts, so look at

\[
f \text{ cts} \implies \lim_{x \to a} f(g(x)) = f( \lim_{x \to a} g(x) )
\]
Find the value of $k$ that makes

$$f(x) = \begin{cases} 
  kx^2 & \text{if } x \leq 3 \\
  2x + k & \text{if } x > 3 
\end{cases}$$

continuous at $x = 3$.

1. **y-value?**  
   \[ f(3) = k(3)^2 = 9k \] 

2. **limit?**  
   \[
   \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} kx^2 = 9k \\
   \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x + k) = 6 + k
   \]

   The limits must be equal for $f$ to be continuous.

   \[ 9k = 6 + k \]

   \[ 8k = 6 \]

   \[ k = \frac{3}{4} \]
Prove that there is at least one real solution to the equation $x^4 + x = 5$. Find an interval of length 1 which contains a solution.

**IVT** If $f$ cts on $[a, b]$ and $N$ is between $f(a)$ and $f(b)$, then there is a solution to $f(x) = N$.

Let $a = 0$ \hspace{1 cm} $f(0) = 0$

$\hspace{1 cm} b = 10 \hspace{1 cm} f(10) = 10010$

$f$ cts since it is a polynomial

$0 < 5 < 10010$

$\therefore$ by IVT there is a solution to $x^4 + x = 5$ (between 0 and 10)

Better: \hspace{1 cm} $a = 1 \hspace{1 cm} f(1) = 2$

$b = 2 \hspace{1 cm} f(2) = 18$ (Solution is between 1 and 2)

Continuing with calc:

$f(1.5) \approx 6.525$ (Solution is between 1 and 1.5)

$f(1.25) \approx 3.6...$ (Solution is between 1.25 and 1.5)

Process is called Bisection Method