2.5-Continuity

Definitions: A function f is *continuous* at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$

f is continuous from the left at $x = a$ if and only if $\lim_{x \to a^-} f(x) = f(a)$

f is continuous from the right at $x = a$ if and only if $\lim_{x \to a^+} f(x) = f(a)$

f is continuous on the interval $[a, b]$ if and only if f cts on (a, b), f cts from the right at $x=a$, f cts from the left at $x=b$.

Definition: f has a *removable discontinuity* at $x = a$ if and only if there is a function g such that g is cts at $x=a$ and $g(x)=f(x)$ for all $x \neq a$.

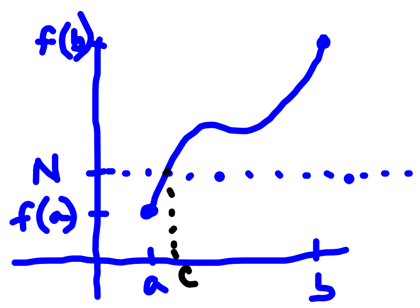
Theorems:

If $\lim_{x \to a} g(x) = b$ and f is continuous at b, then $\lim_{x \to a} f(g(x)) =$
$= f(\lim_{x \to a} g(x)) = f(b)$

Intermediate Value Theorem-

If f cts on $[a,b]$ and N is between $f(a)$ and $f(b)$, then there is a c between a and b such that $f(c) = N$

(i.e. there is a solution to $f(x) = N$)



Thm: polynomials (and most "ordinary" functions) are cts on its domain.

Examples:

Find the values of x for which $f(x) = \dfrac{x^2-9}{x^2-5x+6}$ is not continuous. Determine which, if any, of these discontinuities are removeable.

cts for all x except when $x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$\boxed{x = 2,\ x = 3}$

Vert Asym or Hole in Graph?

$$\lim_{x \to 2} \frac{x^2-9}{x^2-5x-6} = \frac{-5}{0}$$ Vert Asym

Not removeable

$$\lim_{x \to 3} \frac{x^2-9}{x^2-5x-6} \quad \frac{0}{0}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{(x-2)\cancel{(x-3)}} = 6$$

Removeable Discontinuity

$g(3) = 6$

Conc: If limit exists at a discont, then it is a removeable discont

Determine whether the function

$$f(x) = \begin{cases} 2x-3 & \text{if } x \le 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

is continuous at $x = 2$ or not and why. Sketch the graph of the function.

$\lim_{x\to 2} f(x) = f(2)$ 3-point checklist:

① Is $f(2)$ defined? $f(2) = 2(2) - 3 = 1$

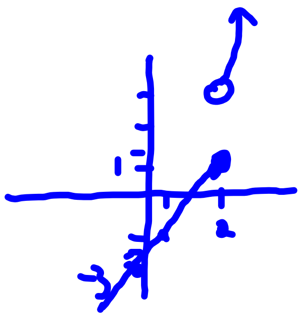
② Does $\lim_{x\to 2} f(x)$ exist?

$(x<2)$ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x-3$
$= 2(2)-3 = 1$
(f cts from left)

$(x>2)$ $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2$
$= 2^2 = 4$

f is not cts at $x=2$ since limit does not exist (left $\neq$ right)

③ Are they equal (limit = y-value)



$f(x)=|x|$

abs val function is cts, so look at

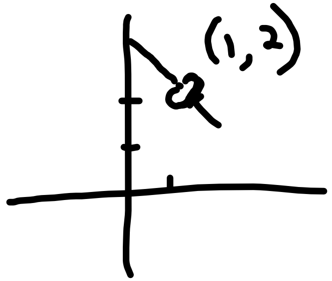
Compute $\lim\limits_{x\to1}\left|\frac{x^2+2x-3}{x^2-1}\right|$

$$= \left|\lim_{x\to1}\frac{x^2+2x-3 \quad 0}{x^2-1 \quad 0}\right|$$

$$= \left|\lim_{x\to1}\frac{(x+3)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}\right|$$

$$= \left|\frac{4}{2}\right| = |2| = \boxed{2}$$

f cts $\rightarrow \lim\limits_{x\to a} f(g(x)) = f(\lim\limits_{x\to a} g(x))$



Find the value of k that makes

$$f(x) = \begin{cases} kx^2 & \text{if } x \leq 3 \\ 2x + k & \text{if } x > 3 \end{cases}$$

continuous at $x = 3$.

① y-value? $f(3) = k(3)^2 = 9k$

② limit? $\lim_{x \to 3^-} f(x) =$ $(x<3)$ $\qquad \lim_{x \to 3^+} f(x) =$ $(x>3)$

$\lim_{x \to 3^-} kx^2 = \underline{9k}$ $\qquad \lim_{x \to 3^+} 2x+k = \underline{6+k}$

must be equal for f to be cts

$9k = 6+k$

$8k = 6$

$\boxed{k = \frac{3}{4}}$

Prove that there is at least one real solution to the equation $x^4 + x = 5$. Find an interval of length 1 which contains a solution.

IVT If f cts $[a,b]$ and N is between $f(a)$ and $f(b)$, then there is a solution to $f(x) = N$.

$a = 0$
$b = 10$
$N = 5$
$f(x) = x^4 + x$

Let $a = 0$ $f(0) = 0$
$b = 10$ $f(10) = 10010$

f cts since it is a polynomial

$0 < 5 < 10{,}010$

$\therefore$ by IVT there is a solution to $x^4 + x = 5$ (between 0 and 10)

Better: $a = 1$ $f(1) = 2$
$b = 2$ $f(2) = 18$ (solution is between 1 and 2)

Continuing w/ calc:
$f(1.5) \approx 6.5625$ (solution is between 1 and 1.5)
$f(1.25) \approx 3.6\ldots$ (solution is between 1.25 and 1.5)
$\vdots$
Process is called Bisection Method