2.7: Tangents, Velocities, and Rates of Change

We are now ready to find a formal way of computing the slope of a line tangent to a curve at a point. Re-view the animation from 2.1 posted on my webpage. What happens as the second $x$ coordinate moves closer to the given tangent-line point?

- the slope of the secant line approaches
- the slope of the tangent line,

Each secant line above passes through the point $(1, 1)$. If $x$ is the $x$-coordinate of the second point, write an expression for the slope of the line between the two points.

$$m_{sec} = \frac{x^2 - 1}{x - 1}$$

Write and solve a limit problem which allows us to find the slope of the tangent line at $x = 1$.

$$m_{tan} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = [2]$$
The function $f(x) = x^2$ is shown on a graph. The slope of the secant line at $x = 1$ and $x = 1.6$ is given by $m_{sec} = \frac{\Delta y}{\Delta x} = \frac{x^2 - 1}{x - 1}$.
More General: Draw any function, a tangent line, and a secant line in the space below. Label the tangent-line point \((a, f(a))\) and the second point on the secant line \((x, f(x))\).

\[
m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}
\]

What should happen as the point \((x, f(x))\) moves closer to the tangent-line point \((a, f(a))\)? Write a limit which explains this mathematically:

\[
m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

Slope of tangent line to \(y = f(x)\) at \(x = a\).
**Most General:** Draw any function, a tangent line, and a secant line again in the space below. Label the tangent-line point \((a, f(a))\) and let \(h\) be the distance between the \(x\) values on the secant line. View the new animation posted under today’s notes for a visual understanding of this.

\[
\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{a+h - a}
\]

\[
m_{\text{sec}} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

Slope of tangent line to \(y = f(x)\) at \(x = a\).
The derivative of a function at \( x = a \) is given by \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \).

**Examples:** Use a limit definition to find the equation of the line tangent to the curve \( f(x) = x^2 + x \) at the point where \( x = 2 \).

**Point, slope**

\[ \text{Point: } x = 2 \quad y = f(2) = 2^2 + 2 = 6 \quad (2, 6) \]

\[ \text{Slope: } \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} \]

\[ = \lim_{x \to 2} \frac{(x^2 + x) - 6}{x - 2} \]

\[ = \lim_{x \to 2} \frac{(x+3)(x-2)}{x - 2} = 5 = m_{+m} \]

**Point-slope**

\[ y - y_1 = m(x - x_1) \]

\[ y - 6 = 5(x - 2) \]

\[ y = 5 \cdot 2 + b \implies b = -4 \]

\[ y = 5x - 4 \]
Use a limit definition to find the derivative of the function $f(x) = \frac{1}{2x+1}$. Compute the slopes of the lines tangent to this graph at $x = 0$, $x = 1$, and $x = -2$.

\[
 f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \quad \text{General case}
\]

\[
 = \lim_{h \to 0} \frac{1}{2(a+h)+1} \cdot \frac{1}{2a+1} \quad \text{Common denominator}
\]

\[
 = \lim_{h \to 0} \frac{1}{h} \left[ \frac{(2a+2h+1)(2a+1) - (2a+1)(2a+1)}{(2a+2h+1)(2a+1)} \right]
\]

\[
 = \lim_{h \to 0} \frac{-2}{(2a+2h+1)(2a+1)} = \frac{-2}{(2a+1)^2}
\]

$x = 0$: $m = f'(0) = \frac{-2}{(2(0)+1)^2} = \frac{-2}{1} = -2$

$x = 1$: $m = f'(1) = \frac{-2}{(2(1)+1)^2} = \frac{-2}{9}$

$x = -2$: $m = f'(-2) = \frac{-2}{(2(-2)+1)^2} = \frac{-2}{9}$
Secant and Tangent Vectors—an Introduction

Illustration of Velocity Vectors (Secant and Tangent):

\[ \vec{r}(t) = \text{position function} \]

\[ \vec{v}_\text{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}(a+h) - \vec{r}(a)}{h} \]

\[ \vec{v}_\text{inst} = \lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h} \]

\[ \vec{r}'(a) = \lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h} \]

\[ \Delta t \quad \lim_{t \to a} \frac{\vec{r}(t) - \vec{r}(a)}{t-a} \]

Secant Vector

\[ \vec{r}(a+h) \cdot \vec{r}(t) \]

\[ t = a \]

\[ h = t - a \]

\[ z = (a+h) \]

\[ \vec{r}(a+h) - \vec{r}(a) \]
Example: Find a vector tangent to the curve \( \mathbf{r}(t) = (3t^2 + 4)i + \sqrt{t}j \) at the point \((7,1)\). Then find parametric equations of the line tangent to the curve at this point.

\[
\mathbf{r}'(a) = \lim_{t \to a} \frac{\mathbf{r}(t) - \mathbf{r}(a)}{t - a}
\]

Find \( t \):
\[
3t^2 + 4 = 7 \quad \sqrt{t} = 1
\]

\[
t = \frac{\sqrt{7} - 1}{2} \quad \mathbf{r}'(a)
\]

Must solve both

\[
\lim_{t \to 1} \frac{(3t^2 + 4)t + \sqrt{t}j - (7t + 1)j}{t - 1}
\]

\[
= \lim_{t \to 1} \frac{(3t^2 - 3)t + (\sqrt{t} - 1)j}{t - 1}
\]

\[
= \lim_{t \to 1} \left( \frac{3t^2 - 3}{t - 1} \right) t + \left( \frac{\sqrt{t} - 1}{t - 1} \right) j
\]

\[
= \left( \lim_{t \to 1} \frac{3(t^2 - 1)}{t - 1} \right) t + \left( \lim_{t \to 1} \frac{\sqrt{t} - 1}{t - 1} \right) j
\]

\[
= \left( \lim_{t \to 1} \frac{3(t^2 - 1)}{t - 1} \right) t + \left( \lim_{t \to 1} \frac{\sqrt{t} - 1}{t - 1} \right) j
\]

\[
= \left( \lim_{t \to 1} \frac{3(t - 1)(t + 1)}{t - 1} \right) t + \left( \lim_{t \to 1} \frac{t - 1}{(t - 1)(t + 1)} \right) j
\]

\[
= 6t + \frac{1}{2}j
\]

Parameter:
\[
\mathbf{r}'_0(t) = 7i + 1j \text{ cont to } (7,1) \text{ on line}
\]

\[
\mathbf{r}(t) = (7t + 1)i + t(1 + \frac{1}{2})j
\]

\[
= (7 + 6t)i + (1 + \frac{1}{2}t)j
\]

\[
x = 7 + 6t \quad y = 1 + \frac{1}{2}t
\]