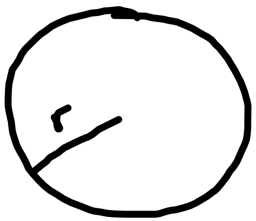
3.10-Related Rates

Idea: As certain quantities change over time, quantities which are related to them (usually via a formula) also change over time.

Example: Oil spilled from a broken tanker spreads in a circular pattern whose radius increases at a constant rate of 0.6 m/sec. How fast is the area of the spill increasing when the radius is 10 m?

$\frac{d}{dt}$



$$\frac{dr}{dt} = 0.6\,\frac{m}{s}$$

$$\frac{dA}{dt} = ?$$

$$r = 10\,m$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(10\,m)\left(0.6\,\frac{m}{s}\right)$$

$$\boxed{\frac{dA}{dt} = 12\pi\,\frac{m^2}{s}}$$

Formulas to know:

$$A_{rect} = lw$$

$$A_{\Delta} = \frac{1}{2}bh$$

$$A_{circ} = \pi r^2$$

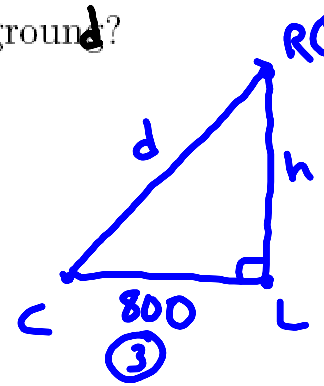
$$V_{prism} = Bh$$

Problem Solving Strategies: (p 217)

1. Read entire problem first and understand what's happening
2. If necessary, draw a diagram
3. * Label your <u>constants</u> and <u>variables</u> in the diagram
4. List all given values and rates and unknown rate.
5. Find an equation that relates your variables
6. If necessary, eliminate unknown variables by relating them to known
7. Differentiate $\left(\frac{d}{dt}\right)$, substitute, and solve

Examples:

1 A camera is positioned 800 m from a rocket launch pad. If the rocket rises vertically at 300 m/sec, how fast is the distance from the camera to the rocket changing when the rocket is 1000 m above ground?



4

$$\frac{dh}{dt} = 300\,\frac{m}{s}$$

$$\frac{dd}{dt} = ?$$

$$h = 1000\text{ m}$$

$$d = \sqrt{800^2 + 1000^2} \approx 1280.625\text{ m}$$

5 Pyth Thm

$$800^2 + h^2 = d^2$$

6 not necessary

7

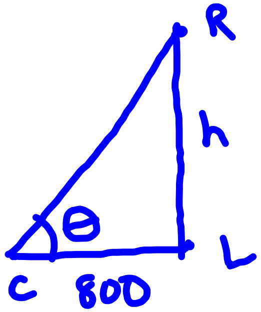
$$2h\frac{dh}{dt} = 2d\frac{dd}{dt}$$

$$2(1000\text{ m})\left(300\,\tfrac{m}{s}\right) = 2(1280.625\text{ m})\frac{dd}{dt}$$

$$\frac{dd}{dt} = \frac{2(1000)(300)}{2(1280.625)}\ \frac{m}{s}$$

$$\approx \boxed{234\,\frac{m}{s}}$$

In the example above, how fast is the angle of elevation of the camera changing at the same instant?



$h = 1000$ m

$\frac{dh}{dt} = 300 \frac{m}{s}$

$\frac{d\theta}{dt} = ?$

$\frac{opp}{adj}$ $\tan\theta = \frac{h}{800} = \frac{1}{800}h$

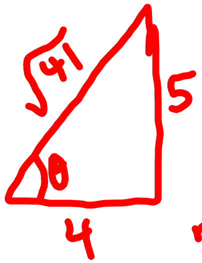
no elim vars

$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{800}\frac{dh}{dt}$

$\frac{d\theta}{dt} = \frac{1}{800}\cdot(300)\cdot\cos^2\theta$

$\frac{d\theta}{dt} = \frac{3}{\cancel{8}}\cdot\frac{\cancel{16}^{2}}{41}$

$\boxed{\frac{d\theta}{dt} = \frac{6}{41}\ \frac{rad}{sec}}$



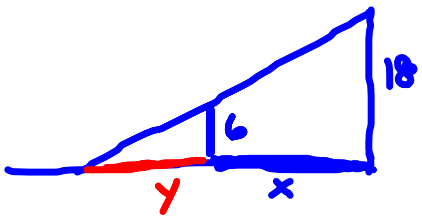
$\cos\theta = \frac{4}{\sqrt{41}}$

ratio of sides

A man 6 feet tall is walking at the rate of 3 ft/sec toward a streetlight 18 ft high.

a) How fast is length of his shadow changing when he is 12 feet from the light?

b) How fast is the tip of his shadow moving at that instant?



$\frac{dx}{dt} = -3 \frac{ft}{s}$

$\frac{dy}{dt} = ?$

$x = 12 ft$

Similar $\triangle$

a) $\frac{y}{6} = \frac{x+y}{18}$ can solve for y

$18y = 6x + 6y$

$12y = 6x$

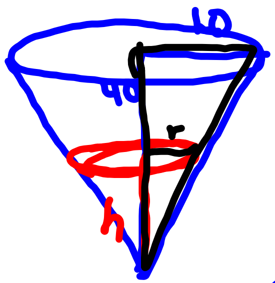
$y = \frac{1}{2}x$

$\frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt}$

$\frac{dy}{dt} = \frac{1}{2}(-3) = \boxed{\frac{-3}{2} \frac{ft}{s}}$

b) $\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt} = -3 + \frac{-3}{2} = \boxed{\frac{-9}{2} \frac{ft}{s}}$

A liquid is to be poured through a conical filter which is 40 cm tall and has a radius of 10 cm at the top. If the liquid is flowing out of the cone at a rate of 5 cm^3/min, how fast is the depth of the liquid changing at that instant? when the water is 10 cm deep?



$\frac{dV}{dt} = -5 \ \frac{\text{cm}^3}{\text{min}}$

$\frac{dh}{dt} = ?$

$h = 10 \text{ cm}$

6 No r

Use similar △s

$\frac{h}{r} = \frac{40}{10}$

$10h = 40r$

$r = \frac{1}{4}h$

Given vol:

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{1}{4}h\right)^2 h$

$V = \frac{1}{48}\pi h^3$

$\frac{dV}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt}$

$-5 = \frac{1}{16}\pi (10)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{-5 \cdot 16}{100\pi} = \boxed{\frac{-4}{5\pi} \ \frac{\text{cm}}{\text{min}}}$