3.11-Linear Approximation and Differentials

Purpose: To understand differentials and linear approximations to a function near a certain point.

Definitions: Given $y = f(x)$, the *differential* dx represents an independent quantity (a small change in x). Then the *differential* dy is given by: $dy = f'(x)\,dx$

Δx

Seemingly Unrelated Topic: Recall graphing $y = \sin x$. What happened as you zoom in on the point corresponding to $x = 0$? The graph of $y = \sin x$ looked like its tangent line at $x = 0$ $(y = x)$

Idea: The tangent line approximates the curve $y = f(x)$ near $x = a$.

What is the equation of the line tangent to $y = f(x)$ at the point where $x = a$?

Point: $x = a$ $y = f(a)$

Slope: deriv $= f'(x)$

$m = f'(a)$

Equation: $y - f(a) = f'(a)(x - a)$

OR $\boxed{y = f(a) + f'(a)(x-a)}$

Definition: The *Linear Approximation* (or *Linearization*) of f at $x = a$ is

$$L(x) = f(a) + f'(a)(x-a)$$

The Connection:

$f(x) \approx L(x) = f(a) + f'(a)(x-a)$ near $x=a$

$\Delta y = f(x) - f(a)$ approximate using $L(x)$

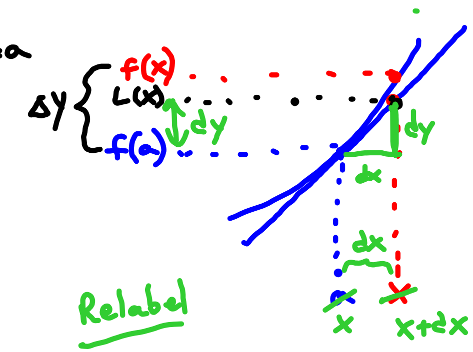
$\approx L(x) - f(a)$

$= [\cancel{f(a)} + f'(a)(x-a)] - \cancel{f(a)}$

$= f'(a)(x-a)$ $\quad x-a = \Delta x = dx$

$= f'(a)\,dx$

$\Delta y \approx dy = f'(x)\,dx$



Relabel

Slope of line $= \frac{dy}{dx}$

$f'(x) = \frac{dy}{dx}$ so

$dy = f'(x)\,dx$

Examples:

Given $y = \sqrt{x}$, find Δy and dy if $x = 4$ and $\Delta x = dx = 1$.

$dy = f'(x)dx$ $\quad x = 4$, $dx = 1$

$f(x) = \sqrt{x}$ so $f'(x) = \frac{1}{2\sqrt{x}}$

$dy = \frac{1}{2\sqrt{x}}\,dx$

$= \frac{1}{2\sqrt{4}} \cdot 1$

$\boxed{dy = \frac{1}{4}}$

$x + \Delta x$
$4+1$

$\Delta y = f(5) - f(4) = \sqrt{5} - \sqrt{4}$

$\approx .23606$

Using differential,

$\sqrt{5} \approx 2 + \frac{1}{4}$

$= \boxed{2.25}$

Use differentials to approximate $\cos 62°$

$dy = f'(x)dx$

$x = 60° = \frac{\pi}{3}$

$dx = 2° = \frac{\pi}{90}$

(NOTE: for $\cos 58°$, $dx = -\frac{\pi}{90}$)

$f(x) = \cos x$

$dy = -\sin x\, dx$

$= -\left(\sin\frac{\pi}{3}\right)\cdot\frac{\pi}{90}$

$= -\frac{\sqrt{3}}{2}\cdot\frac{\pi}{90} = \frac{-\sqrt{3}\pi}{180}$

$\cos 62° \approx \cos\frac{\pi}{3} + \left(-\frac{\sqrt{3}\pi}{180}\right)$

$= \frac{1}{2} - \frac{\sqrt{3}\pi}{180} \approx .46977$

on calculator: $\cos 62° \approx .4694$7

NOTE: Can solve using Linear Approx

$L(x) = f(a) + f'(a)(x-a)$

$a = \pi/3$

$x = 62° = \frac{31\pi}{90}$

$f(x) = \cos x$

$L(x) = \cos\frac{\pi}{3} + \left(-\sin\frac{\pi}{3}\right)\left(\frac{31\pi}{90} - \frac{\pi}{3}\right)$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{90}\right)$

$= \frac{1}{2} - \frac{\sqrt{3}\pi}{180}$

$L(x)$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = \frac{9}{4}$ and use it to approximate $\sqrt{2}$.

$x = 2$ Not $\sqrt{2}$!!

$a = \frac{9}{4}$

$f(x) = \sqrt{x}$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = \sqrt{\frac{9}{4}} + \frac{1}{2\sqrt{\frac{9}{4}}}\left(x - \frac{9}{4}\right)$

$= \frac{3}{2} + \frac{1}{\cancel{2}\cdot\frac{3}{\cancel{2}}}\left(x - \frac{9}{4}\right)$

$\boxed{L(x) = \frac{3}{2} + \frac{1}{3}\left(x - \frac{9}{4}\right)}$

$L(2) = \frac{3}{2} + \frac{1}{3}\left(2 - \frac{9}{4}\right)$

$= \frac{3}{2} + \frac{1}{3}\left(\frac{-1}{4}\right) = \frac{3}{2} - \frac{1}{12} = \boxed{\frac{17}{12}} \approx 1.4167$

on calc $\sqrt{2} \approx 1.4142$

dC

The circumference around the middle of a sphere is measured to be 40 cm, with a possible error of ± 1 cm. Use differentials to estimate the possible error in the volume of the sphere.

Given: $V = \frac{4}{3}\pi r^3$ $\quad C = 2\pi r$ $\quad 40 = 2\pi r$

$dC = 2\pi\, dr$ $\quad r = \frac{40}{2\pi}$

$\pm 1 = 2\pi\, dr$ $\quad = \frac{20}{\pi}$

$dr = \pm\frac{1}{2\pi}$

Method I

$$dV = 4\pi r^2 dr$$
$$= 4\cancel{\pi}\left(\frac{20}{\pi}\right)^2 \cdot \pm\frac{1}{2\cancel{\pi}}$$
$$= \pm\frac{1600}{2\pi^2} = \pm\frac{800}{\pi^2}\text{ cm}^3 \ (\approx 90\text{ cm}^3)$$

Method II

$V(C)$: $\quad r = \frac{C}{2\pi}$

$$V = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3$$
$$V = \frac{\cancel{4}^{1}}{3}\cancel{\pi}\,\frac{C^3}{\cancel{8}_{2}\pi^2} = \frac{1}{6\pi^2}C^3$$
$$dV = \frac{1}{2\pi^2}\underset{40}{C^2}\,\underset{\pm 1}{dC} = \pm\frac{800}{\pi^2}$$

Relative Error:

$$\frac{dV}{V} = \frac{\pm\frac{\cancel{800}^{20}}{\cancel{\pi^2}}}{\frac{1}{\cancel{6}_{3}\cancel{\pi^2}}\cdot 40^{\cancel{3}\,\cancel{2}\,1}} = \frac{3}{40} \approx \pm 7.5\,\%$$

A Better Approximation:

Quadratic Approx of f at x=a

$$\boxed{Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$

Why better?

$f(a) = L(a)$

$f'(a) = L'(a)$

$f(a) = Q(a)$

$f'(a) = Q'(a)$

$f''(a) = Q''(a)$