3.12-Newton’s Method

Recall: In 2.5, we introduced the Bisection Method for solving equations. This method is long and tedious. Here we introduce another method for solving equations.

The *Linear Approximation* of $f$ at $x_0$ is given by

$$L(x) = f(a) + f'(a)(x-a)$$

Set this equal to 0 and solve for $x$. What do you obtain?

$$L(x) = 0 = f(a) + f'(a)(x-a)$$

$$-f(a) = f'(a)(x-a)$$

$$-f(a) = f'(a)(x-a)$$

$$\frac{-f(a)}{f'(a)} = x-a$$

$$x = a - \frac{f(a)}{f'(a)}$$

**Newton’s Method**

To solve $f(x) = 0$ given a starting value $x_0$, create a sequence $\{x_1, x_2, x_3, \ldots \}$ using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which converge to the solution.
Example: Given \( x_0 = 1 \) is an approximate solution of \( x^2 = 2 \), find the solution to 4 decimal places using Newton’s Method.

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

\[ x^2 - 2 = 0 \]
\[ f(x) = x^2 - 2 \]
\[ f'(x) = 2x \]

\[
x_0 = 1
\]
\[
x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^2 - 2}{2 \cdot 1} = 1 - \frac{-1}{2} = 1.5
\]
\[
x_2 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{1.5^2 - 2}{2 \cdot 1.5} = 1.5 - \frac{0.25}{3} = 1.4166667
\]
\[
x_3 = 1.4166667 - \frac{f(1.4166667)}{f'(1.4166667)} \approx 1.414216
\]
\[
x_4 = 1.414216 - \frac{f(1.414216)}{f'(1.414216)} \approx 1.414214
\]

Compare to previous answer.
(Note: Compare with Bisection Method in $x^2 = 2$ below)

$1 < x < 2 \quad f(1.5) = 2.25$

$1 < x < 1.5 \quad f(1.25) = 1.5625$

$1.25 < x < 1.5 \quad f(1.375) = 1.890625$

$1.375 < x < 1.5 \quad f(1.4375) \approx 2.0464$

$1.375 < x < 1.4375 \quad f(1.40625) \approx 1.9775$

$1.40625 < x < 1.4375 \quad f(1.4140625) \approx 2.0228$

$1.40625 < x < 1.421875 \quad f(1.4140625) \approx 1.9195$

$1.4140625 < x < 1.421875 \quad f(1.4145625) \approx 2.0166$

$1.4140625 < x < 1.41796875 \quad f(1.414015625) \approx 2.00525$

$1.4140625 < x < 1.416015625 \quad f(1.41450390625) \approx 2.00231$

$1.4140625 < x < 1.4150390625 \quad f(1.41455078125) \approx 2.00075$

$x \approx 1.41$ (still only 2 decimal place accuracy!)