3.1: The Derivative

Using $\lim_{x \to a} \dfrac{f(x)-f(a)}{x-a}$ or $\lim_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$

Now that we can find the slope of the line tangent to a curve at any point (provided the limit of the slopes exists), we can talk about a new function based on this calculation.

Definition: The *derivative function* of a function f (or the *derivative of* f) is a function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Input $\qquad\qquad\qquad$ Output

$$x \longrightarrow \boxed{f'} \longrightarrow f'(x)$$

Slope of tangent line of f at x.

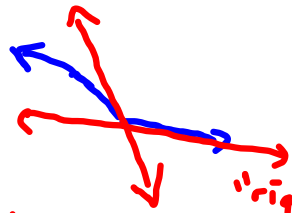
(or slope of graph of f at $x=a$)

Domain: all x such that the limit exists

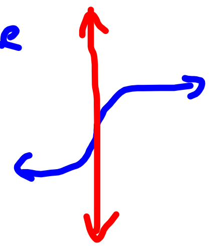
When is f not *differentiable*? (i.e., when does $f'(x)$ not exist or when is x not in the domain of f'?

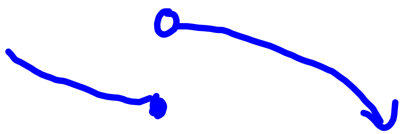
1. When $f(x)$ does not exist
2. When limit DNE
   a. left $\neq$ right
   b. limit is $\pm\infty$
3. When f not cts

Sharp corner



Vertical Tangent Line





Examples:

using the definition

Let $f(x) = \dfrac{1}{\sqrt{1+x}}$. Find $f'(x)$ and use it to determine the slope of the line tangent to f at $x = 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{1+(x+h)}} - \dfrac{1}{\sqrt{1+x}}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{1+x+h}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}} - \frac{1}{\sqrt{1+x}}\cdot\frac{\sqrt{1+x+h}}{\sqrt{1+x+h}} \right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{1+x} - \sqrt{1+x+h}}{\sqrt{1+x+h}\cdot\sqrt{1+x}} \right)\frac{\left(\sqrt{1+x} + \sqrt{1+x+h}\right)}{\left(\sqrt{1+x} + \sqrt{1+x+h}\right)}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{(\cancel{1}+\cancel{x}) - (\cancel{1}+\cancel{x}+h)}{\sqrt{1+x+h}\cdot\sqrt{1+x}\cdot\left(\sqrt{1+x} + \sqrt{1+x+h}\right)} \right)$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\cdot\frac{-\cancel{h}}{\left(\sqrt{1+x+h}\cdot\sqrt{1+x}\right)\left(\sqrt{1+x} + \sqrt{1+x+h}\right)}$$

$$h \to 0$$

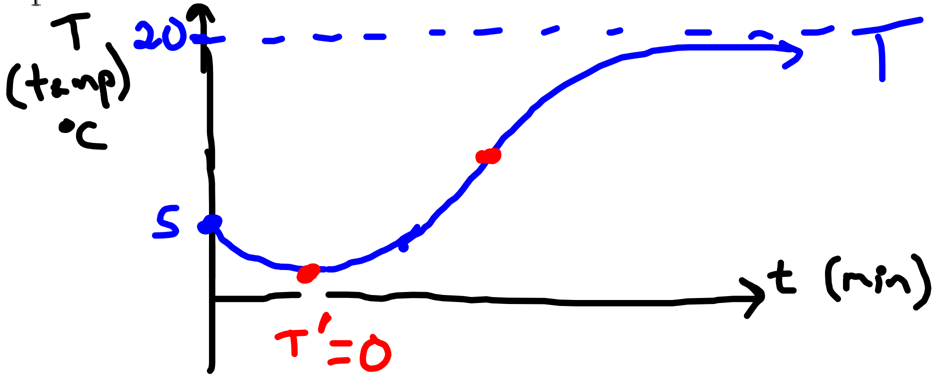
$$= \boxed{\frac{-1}{\left(\sqrt{1+x}\right)\cdot\left(\sqrt{1+x}\right)\cdot\underbrace{\left(\sqrt{1+x} + \sqrt{1+x}\right)}_{2\sqrt{1+x}}}}$$

$$= \boxed{\frac{-1}{2(1+x)\sqrt{1+x}} = -\frac{1}{2}(1+x)^{-3/2}}$$

$$m_{tan} = f'(0) = \boxed{\frac{-1}{2}}$$

Ice cubes are placed in a glass, then at time $t = 0$ the glass is filled with water from a $5°C$ refrigerator. If the glass sits on a table in a $20°C$ room, sketch a rough graph of the temperature of the water, in degrees Celsius, over time. Then sketch the graph of the rate of change of the temperature of the water over time.

derivative



Determine whether the function below is differentiable at $x = 1$:

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$$

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1}$$

$x=1$ use top piece!

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1}$$

$(x<1)$

$$= \lim_{x \to 1^-} \frac{x^2 - 1}{x-1}$$

$$= \lim_{x \to 1^-} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$

$$= 2$$

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1}$$

$(x>1)$

$$\lim_{x \to 1^+} \frac{(2x\cancel{+1}) - \cancel{1}}{x-1} \qquad \frac{2}{0}$$

$$= +\infty$$

$\therefore f'(1)$ DNE or

f not diff at $x=1$