3.4: Limits and Derivatives of Trig Functions

(*NOTE*: we will assume without proof that the functions $f(x) = \sin x$ and $g(x) = \cos x$ are continuous.)

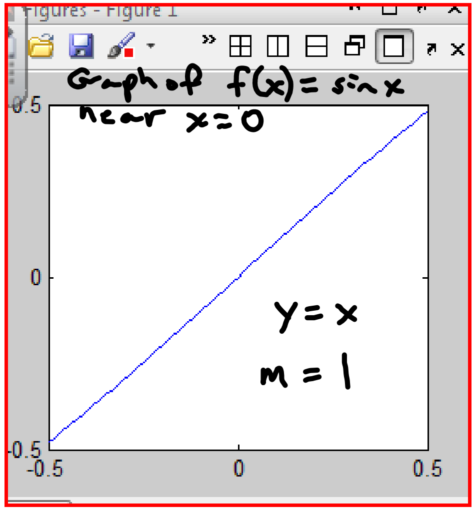
Key Limit: $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

"Proof": $f(x)$ "looks like" tangent line at $x=0$,

$f'(0) = 1$ $\qquad \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

$$\lim_{x\to 0} \frac{\sin x - 0}{x-0} = 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$



Key Limit: $\lim_{x\to 0} \frac{(\cos x - 1)(\cos x + 1)}{x\,(\cos x + 1)} =$

$\sin^2 x + \cos^2 x = 1$

$\cos x = \pm\sqrt{1-\sin^2 x}$

square root → use conjugate

$\cos^2 x - 1 = -\sin^2 x$

Proof: $= \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}$

$$= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x+1)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{-\sin 0}{\cos 0 + 1}$$

$$= 1 \cdot \frac{0}{2} = 0$$

$$\boxed{\lim_{x\to 0} \frac{\cos x - 1}{x} = 0}$$

not required on quiz/exam

We can use these limits to find the derivative of $f(x) = \sin x$ using the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \frac{\sin h}{h}$$

$$\frac{\cos h - 1}{h} \to 0 \qquad \frac{\sin h}{h} \to 1$$

Apply key Limits

$$f'(x) = \sin x(0) + \cos x(1) = \cos x$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Similarly, we can show that $\dfrac{d}{dx}(\cos x) = -\sin x$

Once we know these, we can find the derivatives of all the other trig functions using quotient rules:

Example: $\dfrac{d}{dx}(\tan x) = \dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right)$ HI LO

$$= \frac{(\cos x)(\cos x) + (\sin x)(+\sin x)}{\cos^2 x}$$ SIMPLIFY

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

Summary: Key limits: $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

Key derivatives:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Examples:

$$\lim_{x\to 0}\frac{\tan 7x}{\sin 3x} = \lim_{x\to 0}\frac{\sin 7x}{\cos 7x}\cdot\frac{1}{\sin 3x}$$

$$= \lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\frac{1}{\frac{\sin 3x}{3x}}\cdot\frac{1}{\cos 7x}\cdot\frac{7x}{3x}$$

$$\frac{1}{\frac{\sin 3x}{3x}} = \frac{3x}{\sin 3x}$$

$$= \lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\lim_{x\to 0}\frac{1}{\frac{\sin 3x}{3x}}\cdot\lim_{x\to 0}\frac{1}{\cos 7x}\cdot\lim_{x\to 0}\frac{7\cancel{x}}{3\cancel{x}}$$

$$= 1\cdot\frac{1}{1}\cdot\frac{1}{\underbrace{\cos(7\cdot 0)}_{=1}}\cdot\frac{7}{3}$$

$$= \boxed{\frac{7}{3}}$$

Product Rule

Differentiate: a) $f(x) = x^2 \tan x$

$$f'(x) = \boxed{x^2 \sec^2 x + 2x \tan x}$$

Alt

b) $y = \dfrac{1 - \cos x}{\sin x}$ hi / lo $= \csc x - \cot x$

$$y' = \boxed{\frac{(\sin x)(\sin x) - (1 - \cos x)(\cos x)}{\sin^2 x}}$$

Simplify

$$= \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x} = \frac{1 - \cos x}{\sin^2 x}$$

Find all $a \in [0, 2\pi]$ such that the line tangent to $f(x) = \sin^2 x + \cos x$ at $x = a$ is horizontal.

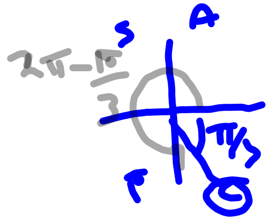
$f' = 0$

$$f'(x) = 2 \sin x \cos x - \sin x = 0$$

$$\sin x \, (2 \cos x - 1) = 0$$

$\sin x = 0$ $\qquad$ $\cos x = \frac{1}{2}$

$x = 0, \pi, 2\pi$ $\qquad$ $x = \frac{\pi}{3}, \frac{5\pi}{3}$



You already know the identity $\sin(2x) = 2\sin x \cos x$. What do you obtain when you differentiate the right hand side of this identity?

Product Rule

$$\frac{d}{dx}\left(2\sin x \cos x\right)$$

$$= (2\sin x)(-\sin x) + (2\cos x)(\cos x)$$ SIMPLIFY

$$= 2(\cos^2 x - \sin^2 x)$$

$$= 2\cos 2x$$

So $\frac{d}{dx}(\sin 2x) = \underline{2}\cos 2x$?

?

Lecture Quiz 2

(No notes, no calculator, show all work to receive credit)

Which statement is true about the function $f(x) = \begin{cases} (x-3)^2+3 & \text{if } x<4 \\ 0 & \text{if } x=4 \\ 2x-4 & \text{if } x>4 \end{cases}$

Support your answer with sufficient work and explanation!

a) f is not continuous at x=4 and does not have a removeable discontinuity

b) f has a removeable discontinuity at x=4

c) f is continuous only from the left at x=4

d) f is continuous only from the right at x=4

e) f is continuous at x=4

$f(4) = 0$

$\lim_{x\to 4^-} f(x) = 4$ $\lim_{x\to 4^+} f(x) = 4$

$\therefore \lim_{x\to 4} f(x) = 4$

$0 \neq 4$