3.4: Limits and Derivatives of Trig Functions

(NOTE: we will assume without proof that the functions $f(x) = \sin x$ and $g(x) = \cos x$ are continuous.)

Key Limit: \[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

"Proof": $f(x)$ "looks like" tangent line at $x=0$.

\[
f'(0) = 1 = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0}
\]

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

Key Limit: \[
\lim_{x \to 0} \frac{\cos x - (\cos x + 1)}{x} = \frac{\sin^2 x}{(\cos x + 1)}
\]

Proof: \[
\lim_{x \to 0} \frac{\cos^3 x - 1}{x(\cos x + 1)}
\]

\[
= \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)}
\]

\[
= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}
\]

\[
= 1 \cdot \frac{0}{2} = 0
\]

\[
\lim_{x \to 0} \frac{\cos x - 1}{x} = 0
\]
We can use these limits to find the derivative of $f(x) = \sin x$ using the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cosh - \sin x + \cos x \sinh}{h}$$

$$= \lim_{h \to 0} \frac{\sin x (\cosh - 1) + \cos x \cdot \frac{\sin h}{h}}{h}$$

Apply Key Limits

$$f'(x) = \sin x (0) + \cos x (1) = \cos x$$

$$\frac{d}{dx} (\sin x) = \cos x$$
Similarly, we can show that \( \frac{d}{dx} (\cos x) = -\sin x \)

Once we know these, we can find the derivatives of all the other trig functions using quotient rules:

Example: \( \frac{d}{dx} (\tan x) = \frac{\frac{d}{dx} (\sin x)}{\cos x} \)

\[
= \frac{(\cos x)(\cos x) + (\sin x)(\sin x)}{\cos^2 x} \quad \text{Simplify}
\]

\[
= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}
\]

\[
= \frac{1}{\cos^2 x} = \sec^2 x \quad \frac{d}{dx} (\tan x) = \sec^2 x
\]

Summary: Key limits: \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \)

Key derivatives:

\[
\frac{d}{dx} (\sin x) = \cos x
\]

\[
\frac{d}{dx} (\cos x) = -\sin x
\]

\[
\frac{d}{dx} (\tan x) = \sec^2 x \quad \frac{d}{dx} (\sec x) = \sec x \tan x
\]

\[
\frac{d}{dx} (\cot x) = -\csc^2 x \quad \frac{d}{dx} (\csc x) = -\csc x \cot x
\]
Examples:

\[
\lim_{x \to 0} \frac{\tan 7x}{\sin 3x} = \lim_{x \to 0} \frac{\sin 7x}{\cos 7x} \cdot \frac{1}{\sin 3x} = \lim_{x \to 0} \frac{\sin 7x}{7x} \cdot \frac{1}{\sin 3x} \cdot \frac{1}{\cos 7x} \cdot \frac{7x}{3x} = 1 \cdot \frac{1}{1} \cdot \frac{1}{\cos(0)} \cdot \frac{7}{3} = \frac{7}{3}
\]
Differentiate: a) \( f(x) = x^2 \tan x \)

\[
f'(x) = x^2 \sec^2 x + 2x \tan x
\]

b) \( y = \frac{1 - \cos x}{\sin x} \)

\[
y' = \frac{(\sin x)(\sin x) - (1-\cos x)(\cos x)}{\sin^2 x}
\]

Simplify:

\[
y' = \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x} = \frac{1 - \cos x}{\sin^2 x}
\]

Find all \( a \in [0, 2\pi] \) such that the line tangent to \( f(x) = \sin^2 x + \cos x \) at \( x = a \) is horizontal.

\[
f''(x) = 2 \sin x \cos x - \sin x = 0
\]

\[
\sin x (2\cos x - 1) = 0
\]

\[
\sin x = 0 \quad \cos x = \frac{1}{2}
\]

\( x = 0, \pi, 2\pi \)

\( x = \frac{\pi}{3}, \frac{5\pi}{3} \)
You already know the identity $\sin(2x) = 2 \sin x \cos x$. What do you obtain when you differentiate the right hand side of this identity?

\[
\frac{d}{dx}(2 \sin x \cos x) = (2 \sin x)(-\sin x) + (2 \cos x)(\cos x)
\]

Simplify

\[
= 2(\cos^2 x - \sin^2 x)
\]

\[
= 2 \cos 2x
\]

So \[
\frac{d}{dx}(\sin 2x) = 2 \cos 2x \quad ?
\]
Lecture Quiz 2

(No notes, no calculator, show all work to receive credit)

Which statement is true about the function

\[
f(x) = \begin{cases} 
(x-3)^2 + 3 & \text{if } x < 4 \\
0 & \text{if } x = 4 \\
2x - 4 & \text{if } x > 4 
\end{cases}
\]

Support your answer with sufficient work and explanation!

a) \( f \) is not continuous at \( x=4 \) and does not have a removable discontinuity

b) \( f \) has a removable discontinuity at \( x=4 \)

c) \( f \) is continuous only from the left at \( x=4 \)

d) \( f \) is continuous only from the right at \( x=4 \)

e) \( f \) is continuous at \( x=4 \)

\[
\begin{align*}
f(4) &= 0 \\
\lim_{{x \to 4^-}} f(x) &= 4 \\
\lim_{{x \to 4^+}} f(x) &= 4 \\
\therefore \lim_{{x \to 4}} f(x) &= 4 \\
0 &\neq 4
\end{align*}
\]