3.5: The Chain Rule

We know $\dfrac{d}{dx}(x^2) = 2x$. Does $\dfrac{d}{dx}((x^2+4)^2) = 2(\underline{x^2+4})$?
(HINT: Expand the function, then differentiate).

$$(x^2+4)^2 = x^4 + 8x^2 + 16$$

$$\frac{d}{dx}\left((x^2+4)^2\right) = 4x^3 + 16x$$

$$= 4x(\underline{x^2+4})$$

$$= \underline{2}(\underline{x^2+4})(2x)$$

Recall: The *composition* of 2 functions f and g is defined by $(f \circ g)(x) = f(g(x))$

Define f and g for the above function.

$$g(x) = x^2 + 4$$

$$f(x) = x^2 \quad \underline{\text{OR}} \quad f(u) = u^2$$

End of 3.4 $\frac{d}{dx}\left(\sin(2x)\right) = 2\cos 2x$

The Chain Rule: If f and g are differentiable functions, $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

An alternate version of the Chain Rule states that * $\boxed{\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)}$ *

Examples:

Find the derivatives of the following:

$f(x) = 4\cos(x^3)$

$f'(x) = \boxed{-4\sin(x^3) \cdot 3x^2}$

$= \boxed{-12x^2 \sin(x^3)}$

u / g(x) / stuff

$y = (\underbrace{x^2 - x + 1}_{})^{23}$

$\boxed{y' = 23(x^2 - x + 1)^{22}(2x - 1)}$

Differentiate the following:

$(\sin x)^2$ Product Rule

$f(x) = x^3 \cdot \sin^2 x$

$$\boxed{f'(x) = x^3(2)(\overset{\text{chain}}{\sin x})(\cos x) + 3x^2 \sin^2 x}$$

Chain Rule

$y = (1 + x^5 \cot x)^{-8}$

Prod Rule

$y' = -8(1 + x^5 \cot x)^{-9}(x^5(-\csc^2 x) + 5x^4 \cdot \cot x)$

Alt: $g(x) = (2x+3)^3 (4x^2-1)^{-8}$

Product

$$g(x) = \frac{(2x+3)^3}{(4x^2-1)^8} \quad \begin{matrix} \text{HI} \\ \text{LO} \end{matrix}$$

$$y = \left(\frac{x-5}{2x+1}\right)^3$$

$$g'(x) = \boxed{\frac{(4x^2-1)^8 (3)(2x+3)^2 (2) - (2x+3)^3 (8)(4x^2-1)^7 (8x)}{(4x^2-1)^{16}}}$$

simplify

$$= \frac{2\cancel{(4x^2-1)^7} (2x+3)^2 \left[(4x^2-1)(3) - (2x+3)(4)(8x)\right]}{(4x^2-1)^{\cancel{16}\, 9}}$$

$$= \frac{2(2x+3)^2 (12x^2 - 3 - 64x^2 - 96x)}{(4x^2-1)^9} \ldots$$

$y = \left(\frac{x-5}{2x+1}\right)^3$ Chain Rule

$$y' = \boxed{3\left(\frac{x-5}{2x+1}\right)^2 \left(\frac{(2x+1)(1) - (x-5)(2)}{(2x+1)^2}\right)}$$

Quot.

Given the table of values below, find the derivative of $f(g(x))$ and $g(f(x))$ at $x = -1$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	2	3	2	-3
2	0	4	1	-5

$$\frac{d}{dx} f(g(x)) = f'(g(x))\,g'(x)$$

at $x=-1$

$$= f'(g(-1))\,g'(-1)$$

$$= f'(2)\,g'(-1)$$

$$= 4 \cdot -3$$

$$= \boxed{-12}$$

$$\frac{d}{dx} g(f(x)) = g'(f(x)) \cdot f'(x)$$

at $x=-1$ $= g'(f(-1)) \cdot f'(-1)$

$$= g'(2) \cdot f'(-1)$$

$$= -5 \cdot 3$$

$$= \boxed{-15}$$