3.6-Implicit Differentiation

Ex $x^2 + y^2 = 1$

The equation $F(x, y) = 0$ *implicitly* defines a relation (not necessarily a function) between y and x. The *graph* of $F(x, y) = 0$ is the set of all points (x, y) such that the equation holds $(\{(x, y)|F(x, y) = 0\})$. Given a graph of an implicitly-defined relation, we can still talk about the slope of the line tangent to the curve at a given point.

Method for Implicit Differentiation:

1. Done when y is not explicitly defined as a function of x.
2. Differentiate both sides of the equation, remembering that y depends on x (can call it $\underline{y(x)}$)
3. Solve for $y'(x)$

Examples:

Find $\frac{dy}{dx}$ implicity if $xy = 1$. Then solve for y and show you get the same answer.

Product Rule

$x \cdot y(x) = 1$

$x \cdot y'(x) + 1 \cdot y(x) = 0$

$xy' + y = 0$

$\boxed{y' = \frac{-y}{x}}$

$y' = \frac{-\frac{1}{x}}{x} = \frac{-1}{x^2}$

$xy = 1$

$y = \frac{1}{x} = x^{-1}$

$\boxed{y' = \frac{-1}{x^2}}$

CANNOT SOLVE EXPLICTLY FOR Y

Find $\dfrac{dy}{dx}$ if $5y^2 + \sin y = x^2$

$5(y(x))^2 + \sin(y(x)) = x^2$

$10\, y(x)\, y'(x) + \cos(y(x)) \cdot y'(x) = 2x$

$10\, y\, y' + \cos y \cdot y' = 2x$

ALGEBRA: multiply everything out, move y' terms on one side, factor and divide

$y'(10y + \cos y) = 2x$

$$\boxed{y' = \frac{2x}{10y + \cos y}}$$

Find the slope of the line tangent to $y^2 - x + 1 = 0$ at the point $(\overset{x}{2}, \overset{y}{-1})$

Explicit:

$y = \sqrt{x-1}$

$y' = \frac{1}{2}(x-1)^{-1/2}$

$x=2$

$y' = \frac{1}{2}(2-1)^{-1/2}$

$= \boxed{\frac{1}{2}}$

IMPLICIT:

$2yy' - 1 = 0$

$x=2$
$y=-1$

$-2y' - 1 = 0$

$\boxed{y' = -\frac{1}{2}}$?

Can substitute first, then solve

PROBLEM

$\sqrt{y^2} = \sqrt{x-1}$

$|y| = \sqrt{x-1}$

$y = \pm\sqrt{x-1}$

Since $y=-1$ at the point,

$y = -\sqrt{x-1}$

Show that the curves $y = 3x^2$ and $x^2 + 2y^2 = 19$ are orthogonal. $\perp$

$y = 3x^2$

$y' = 6x$

$2x + 4yy' = 0$

$y' = \frac{-2x}{4y}$

$y' = \frac{-x}{2y}$

$m_1 = -\frac{1}{m_2}$

(use tangent line slopes; i.e. derivatives)

Method I

find intersections

$y = 3x^2$

$x^2 + 2y^2 = 19$

$x^2 + 2(3x^2)^2 = 19$

$18x^4 + x^2 - 19 = 0$

$(18x^2+19)(x^2-1) = 0$

$\cancel{x^2 = \frac{-19}{18}}$ $\quad$ $x^2 = 1$

$x = \pm 1 \quad y = 3x^2 = 3$

$(-1,3)$ and $(1,3)$

$(1,3)$

$y' = 6$ and $y' = \frac{-1}{2 \cdot 3} = \frac{-1}{6}$ neg recip, so $\perp$

$(-1,3)$

$y' = -6$ and $y' = \frac{1}{6}$

Method II

$y' = 6x \qquad y' = \frac{-x}{2y}$

where curves intersect, $y = 3x^2$

$y' = \frac{-x}{2(3x^2)} = \frac{-1}{6x}$

The equations $x^2 + y^2 = r^2$ and $y = mx$ represent *families of curves* for different constants r and m. Show that these families of curves are orthogonal.

$$2x + 2yy' = 0 \qquad\qquad y = mx$$

$$y' = \frac{-2x}{2y} \qquad\qquad y' = m$$

$$= \frac{-x}{y}$$ $y = mx$, where they intersect

$$y' = \frac{-x}{mx} = \frac{-1}{m}$$ neg recip, $\therefore \perp$

Alt

$y = mx$, so $m = \frac{y}{x}$

So $y' = \frac{-x}{y}$ and $y' = m = \frac{y}{x}$