3.7-Derivatives of Vector Functions

Recall definition:

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\left(x(t+h)\vec{i} + y(t+h)\vec{j}\right) - \left(x(t)\vec{i} + y(t)\vec{j}\right)}{h}$$

subtract by components

$$= \lim_{h\to 0} \frac{\left(x(t+h) - x(t)\right)\vec{i} + \left(y(t+h) - y(t)\right)\vec{j}}{h}$$

Multiply $\frac{1}{h}$ by each component

$$= \lim_{h\to 0} \left[\left(\frac{x(t+h) - x(t)}{h}\right)\vec{i} + \left(\frac{y(t+h) - y(t)}{h}\right)\vec{j}\right]$$

limit of each component

$$\vec{r}\,'(t) = \left(\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}\right)\vec{i} + \left(\lim_{h\to 0} \frac{y(t+h) - y(t)}{h}\right)\vec{j}$$

$$\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j}$$

* If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$,

* then $\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j}$ Deriv of each component

(video of 2.7 for details)

What the derivative of a vector function tells us:

1) a vector tangent to the graph

2) a direction vector of the tangent line $(\vec{r}_0 + t\vec{v})$

3) velocity, if $\vec{r}$ is position

3.8

Acceleration vector- derivative of velocity

derivative of (deriv of position)

<u>second</u> <u>derivative</u> of position

Examples:

x Find the velocity, speed, and acceleration for the curve $\mathbf{r}(t) =< 4\sin t, 4\cos t >$ at the point $(2, -2\sqrt{3})$.

$\vec{v}(t) \perp \vec{r}(t)$ at ALL t!

velocity: $\vec{r}'(t) = (4\cos t)\vec{i} + (-4\sin t)\vec{j}$

$x=2$, not t! – Find t

$4\sin t = 2$ $\qquad$ $4\cos t = -2\sqrt{3}$

$\sin t = \frac{1}{2}$ $\qquad$ $\cos t = -\frac{\sqrt{3}}{2}$

$t = \frac{\pi}{6}, \frac{5\pi}{6}$ $\qquad$ $t = \frac{5\pi}{6}, \frac{7\pi}{6}$

$t = \frac{5\pi}{6}$ satisfies BOTH equations!

NOTE: $\vec{v} \perp \vec{r}$

$\vec{r}'\left(\frac{5\pi}{6}\right) = \left(\cancel{4}^{\,2}\cos\frac{5\pi}{6}\right)\vec{i} + \left(-4\sin\frac{5\pi}{6}\right)\vec{j} = \boxed{-2\sqrt{3}\,\vec{i} - 2\vec{j}}$

$\left(\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}, \ \sin\frac{5\pi}{6} = \frac{1}{2}\right)$

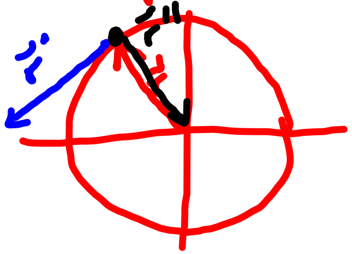
speed: $\left|\vec{r}'\left(\frac{5\pi}{6}\right)\right| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = \boxed{4}$

NOTE: $|\vec{r}'(t)| = \sqrt{(4\cos t)^2 + (4\sin t)^2} = 4$

accel: $\vec{r}''(t) = (-4\sin t)\vec{i} + (-4\cos t)\vec{j}$

$\vec{r}''\left(\frac{5\pi}{6}\right) = \left(-4\sin\frac{5\pi}{6}\right)\vec{i} + \left(-\cancel{4}^{\,2}\cos\frac{5\pi}{6}\right)\vec{j} = \boxed{-2\vec{i} + 2\sqrt{3}\,\vec{j}}$

NOTE: Graph of $\vec{r}$ is circle!



Product Rule!

Find a unit tangent vector for the curve $\mathrm{r}(t) = < t\cos 2t, t\sin 2t >$ at the point where $t = \pi$.

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|}$$

$$\vec{r}\,'(t) = \left(1\cdot\cos 2t + t(-\underline{2}\sin 2t)\right)\vec{i} + \left(1\cdot\sin 2t + t(\underline{2}\cos 2t)\right)\vec{j}$$

Chain Rule

$t = \pi$

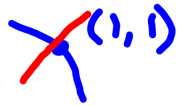
$$\vec{r}\,'(\pi) = (\overset{1}{\cos 2\pi} - 2\pi\overset{0}{\sin 2\pi})\vec{i} + (\overset{0}{\sin 2\pi} + 2\pi\overset{1}{\cos 2\pi})\vec{j}$$

$$= 1\vec{i} + 2\pi\vec{j}$$

$$\vec{T}(\pi) = \frac{\vec{i} + 2\pi\vec{j}}{\sqrt{1+4\pi^2}}$$

Do Scalar Mult!

$$= \boxed{\left(\frac{1}{\sqrt{1+4\pi^2}}\right)\vec{i} + \left(\frac{2\pi}{\sqrt{1+4\pi^2}}\right)\vec{j}}$$



The graphs of $\mathbf{r}_1(t) = t^2\mathbf{i} + t^3\mathbf{j}$ and $\mathbf{r}_2(t) =< \sqrt{2}\cos t, \sqrt{2}\sin t >$ intersect at the point (1,1). Find the angle of intersection to the nearest degree.

Use $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ on the tangent vectors

$\vec{r}_1{}'(t) = 2t\vec{i} + 3t^2\vec{j}$ Find t: $t^2 = 1$, $t^3 = 1$; $t = \pm 1$, $t = 1$

$\vec{a} = \vec{r}_1{}'(1) = 2\vec{i} + 3\vec{j}$ $t = 1$ solves BOTH equations

$\vec{r}_2{}'(t) = (-\sqrt{2}\sin t)\vec{i} + (\sqrt{2}\cos t)\vec{j}$ Find t: $\sqrt{2}\cos t = 1$, $\sqrt{2}\sin t = 1$

$\cos t = \frac{1}{\sqrt{2}}$ $\sin t = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$t = \frac{\pi}{4}$ solves BOTH equations

$\vec{b} = \vec{r}_2{}'\left(\frac{\pi}{4}\right) = \left(-\sqrt{2}\underbrace{\sin\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}}\right)\vec{i} + \left(\sqrt{2}\underbrace{\cos\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}}\right)\vec{j}$

$\vec{b} = -1\vec{i} + 1\vec{j}$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ $\vec{a} = 2\vec{i} + 3\vec{j}$ $\vec{b} = -1\vec{i} + 1\vec{j}$

$= \dfrac{(2)(-1) + (3)(1)}{\sqrt{13}\cdot\sqrt{2}} = \dfrac{1}{\sqrt{26}}$

$\theta = \boxed{\cos^{-1}\left(\frac{1}{\sqrt{26}}\right)} \approx \boxed{79^\circ}$