## 4.1: Exponential Functions

variable in exponent!

**Definition:** An *exponential function* is a function of the form $f(x) = a^x$, $a \neq 1$.

$a > 0$

**Graph and Graphical Properties of $f(x) = a^x$:**

$a > 1$


Domain: $\mathbb{R}$

Range: $(0, \infty)$

Y Intercept $(0,1)$

HA: $y = 0$

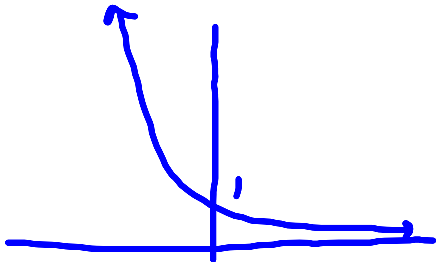
cts

diff

$\lim_{x \to -\infty} a^x = 0$

$\lim_{x \to \infty} a^x = \infty$

$0 < a < 1$




Domain: $\mathbb{R}$

Range: $(0, \infty)$

y int $(0,1)$

HA $y = 0$

cts

diff

$\lim_{x \to \infty} a^x = 0$

$\lim_{x \to -\infty} a^x = \infty$

Properties of exponential functions:

$$a^x \cdot a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{xy}$$

$$(ab)^x = a^x b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x$$

NOTE

$(a \pm b)^x \neq a^x \pm b^x$

$$f(x) = a^x$$

Using the definition of the derivative, we see that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^{0+h} - a^0}{h}$$

looks like defn of deriv with $x=0$

$$= a^x \cdot f'(0)$$

**Definition:** $e$ is the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

$$\therefore \quad \frac{d}{dx}(e^x) = e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

$$* \boxed{\frac{d}{dx}(e^x) = e^x} *$$

$e \approx 2.718281828574\ldots$

*Examples:*

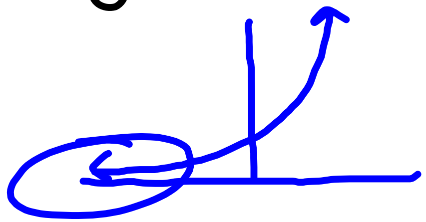
Compute the following limits:

$$\lim_{x\to 0^-} e^{\frac{1}{x}}$$

Since $e^x$ is cts, Look at $\lim_{x\to 0^-} \frac{1}{x}$ $\frac{+}{-}$ $(x<0)$

$= -\infty$

So $\lim_{x\to 0^-} e^{\frac{1}{x}} = 0$



Alt $y=\frac{1}{x}$ as $x\to 0^-$, $y\to -\infty$

$\lim_{y\to -\infty} e^y = 0$

$$\lim_{x\to -\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$$

$* \ \frac{e^{3x}}{e^{-3x}} = e^{(3x - -3x)}$

dominating term as $x\to -\infty$ is $e^{-3x}$

$= \lim_{x\to -\infty} \frac{e^{-3x}(e^{6x} - 1)}{e^{-3x}(e^{6x}+1)}$

$= \lim_{x\to -\infty} \frac{e^{6x} - 1}{e^{6x} + 1} = \boxed{-1}$ ($e^{6x}\to 0$)

$\lim_{x\to 2} 3^{-\frac{x}{(x-2)^2}}$ Look at $\lim_{x\to 2} \frac{-x}{(x-2)^2} \frac{-}{+} = -\infty$

So $\lim_{x\to 2} 3^{-\frac{x}{(x-2)^2}} = 0$

Product Rule

Differentiate $f(x) = e^{ax}\cos(bx)$, where $a$ and $b$ are constants.

$f'(x) = e^{ax}\cdot a\cos(bx) + e^{ax}\cdot(-\sin bx)$

Implicit

Find $y'$ given $e^y - e^{-y} = x$.

$$e^y \cdot y' + e^{-y}(+y') = 1$$

$$y'(e^y + e^{-y}) = 1$$

$$\boxed{y' = \frac{1}{e^y + e^{-y}}}$$

A *differential equation* is an equation involving an unknown function and one or more of its derivatives. Show that the function $y = 2e^{-3x}$ is a solution to the differential equation $y' = -3y$

subs into y

$$(2e^{-3x})' \stackrel{?}{=} -3(2e^{-3x})$$

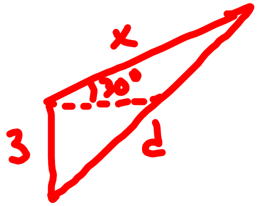
$$2e^{-3x}\cdot(-3)$$

$$-6e^{-3x} \stackrel{\checkmark}{=} -6e^{-3x}$$

**Lecture Quiz 3: DO IN GROUPS (max 4 people/group)!**

**No notes, no calculator. Each group turn in ONE solution (including work) with all group members' names.**

A plane passing 3 miles above a radar station is climbing at a 30° angle from the horizontal and travelling at 480 mi/hr. Find the rate at which the distance from the plane to the station is increasing when the plane has flown 10 miles.
(HINT: The Law of Cosines is $c2 = a2 + b2 - 2\,ab\cos C$




$\frac{dx}{dt} = 480$

$\frac{dd}{dt} = ?$

$x = 10$

$d^2 = 9 + x^2 - 2(3)x\cos 120°$ ($\cos 120° = -\frac{1}{2}$)

$d^2 = 9 + x^2 + 3x$

$2d\frac{dd}{dt} = 2x\frac{dx}{dt} + 3\frac{dx}{dt}$

$d = \sqrt{9 + 10^2 + 30} = \sqrt{139}$

$2\sqrt{139}\,\frac{dd}{dt} = 2(10)(480) + 3(480)$

$\frac{dd}{dt} = \frac{(23)(480)}{2\sqrt{139}}\ \frac{mi}{hr}$