4.1: Exponential Functions

**Definition:** An exponential function is a function of the form \( f(x) = a^x, \ a \neq 1. \)

Graph and Graphical Properties of \( f(x) = a^x: \)

Properties of exponential functions:

\[
\begin{align*}
   a^x \cdot a^y &= a^{x+y} \\
   \frac{a^x}{a^y} &= a^{x-y} \\
   (a^x)^y &= a^{xy} \\
   (ab)^x &= a^x b^x \\
   (\frac{a}{b})^x &= \frac{a^x}{b^x} \\
   a^{-x} &= \frac{1}{a^x} = (\frac{1}{a})^x
\end{align*}
\]

**NOTE:**

\[
(a+b)^x \neq a^x + b^x
\]
\[ f(x) = a^x \]

Using the definition of the derivative, we see that

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h} \\
&= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \\
&= a^x \cdot f'(0)
\end{align*}
\]

**Definition:** \( e \) is the number such that \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \)

\[
\begin{align*}
\therefore \quad \frac{d}{dx}(e^x) &= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\
\therefore \quad \frac{d}{dx}(e) &= e^x \quad \star \star \\
e &\approx 2.718281828574...
\end{align*}
\]
Examples:

Compute the following limits:

\[
\lim_{x \to -0} e^{\frac{1}{x}} \quad \text{since } e^x \text{ is cts,}
\]

Look at \( \lim_{x \to 0^-} \frac{1}{x} \quad (x < 0) \)

\[
= -\infty
\]

So \( \lim_{x \to 0^-} e^{\frac{1}{x}} = 0 \)

\[
\frac{e^{3x}}{e^{-3x}} = e^{3x} \quad \text{dominating term as } x \to -\infty
\]

\[
\lim_{x \to -\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x \to -\infty} \frac{e^{6x} - 1}{e^{6x} + 1} = -1
\]

\[
\overbrace{\text{All } y = \frac{1}{x} \quad \text{as } x \to 0^- \quad y \to -\infty}
\]

\[
\lim_{y \to -\infty} e^y = 0
\]
\[ \lim_{x \to 2} 3 \frac{x}{(x-2)^2} \]

Look at \( \lim_{x \to 2} \frac{-x}{(x-2)^2} + = -\infty \)

So \( \lim_{x \to 2} 3 \frac{-x}{(x-2)^2} = 0 \)

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**Product Rule**

Differentiate \( f(x) = e^{ax} \cos(bx) \), where \( a \) and \( b \) are constants.

\[
 f'(x) = e^{ax} \cdot a \cos(bx) + e^{ax} \cdot (-b \sin bx) 
\]
Find $y'$ given $e^y - e^{-y} = x$.

\[ e^y \cdot y' + e^{-y} (1 + y') = 1 \]

\[ y' (e^y + e^{-y}) = 1 \]

\[ y' = \frac{1}{e^y + e^{-y}} \]
A *differential equation* is an equation involving an unknown function and one or more of its derivatives. Show that the function \( y = 2e^{-3x} \) is a solution to the differential equation \( y' = -3y \)

\[
(2e^{-3x})' = -3(2e^{-3x})
\]

\[
2e^{-3x}(-3) = -6e^{-3x}
\]

\[
-6e^{-3x} = -6e^{-3x}
\]
Lecture Quiz 3: DO IN GROUPS (max 4 people/group)!

No notes, no calculator. Each group turn in ONE solution (including work) with all group members' names.

A plane passing 3 miles above a radar station is climbing at a 30° angle from the horizontal and travelling at 480 mi/hr. Find the rate at which the distance from the plane to the station is increasing when the plane has flown 10 miles.

(HINT: The Law of Cosines is \( c^2 = a^2 + b^2 - 2ab \cos C \))

\[
\frac{dx}{dt} = 480 \\
\frac{dd}{dt} = ? \\
\frac{dx}{dt} = 10 \\
\frac{dd}{dt} = \frac{\sqrt{9 + 10^2 + 360} \cdot \frac{2x}{dt} + 2dx}{dt} = \frac{\sqrt{139} \cdot 2(10)(480)}{2 \sqrt{139} \text{ mi}} \\
= \frac{(23)(480)}{2 \sqrt{139} \text{ mi/hr}}
\]