4.2-Inverse Functions and Their Derivatives

Definitions:

f is a *one-to-one function* if and only if for any y, there is a unique x such that $f(x) = y$.

To show one-to-one: Set $f(a) = f(b)$ and show ONLY solution is $a = b$

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If f is one-to-one, the *inverse* of f is a function f^{-1} such that if $y = f^{-1}(x)$, then $x = f(y)$.

$\cancel{f^{-1} = \frac{1}{f}}$

If (a, b) is on the graph of $y = f(x)$, then (b, a) is on the graph of $y = f^{-1}(x)$.

If f is one-to-one and differentiable at $x = g(a)$ and $g = f^{-1}$, then

$y = g(x)$ means

$x = f(y)$ implicit diff

$1 = f'(y)\, y'$

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(g(x))}$$

so $\boxed{g'(a) = \dfrac{1}{f'(g(a))}}$ ***

Examples:

a) Show $f(x) = \sqrt{3x-2}$ is one-to-one

b) Find the inverse of $f(x) = \sqrt{3x-2}$

a) $f(a) = f(b)$

$(\sqrt{3a-2})^2 = (\sqrt{3b-2})^2$

$3a-2 = 3b-2$

$3a = 3b$

$a = b$ only solution,

so f is one-to-one

b) $y = \sqrt{3x-2}$ switch x and y

$x = \sqrt{3y-2}$ Solve for y

$x^2 = 3y-2$

$x^2+2 = 3y$

$\boxed{f^{-1}(x) = y = \frac{1}{3}(x^2+2)}$

Problem: NOT ONE-TO-ONE

Solution: restrict domain

$\boxed{f^{-1}(x) = \frac{1}{3}(x^2+2)\ ;\ x \geq 0}$*

*Cannot use $x \leq 0$ since domain of f^{-1} must be same as range of f

Note

f^{-1}: D: $[0,\infty)$

R: $[\frac{2}{3},\infty)$

f: D $[\frac{2}{3},\infty)$ since $3x-2 \geq 0$

R $[0,\infty)$

Given $f(x) = \dfrac{3-x}{1-x}$, find f^{-1}

$$\frac{x}{1} = \frac{3-y}{1-y} \quad \text{Solve for } y$$

$$x(1-y) = 3-y$$

$$x - xy = 3 - y$$

$$y - xy = 3 - x$$

$$y(1-x) = 3-x$$

$$\boxed{f^{-1}(x) = y = \frac{3-x}{1-x}}$$

NOTE $f^{-1} = f$!

Given $g(x)$ is the inverse of $f(x) = x + x^2 + e^x$, find $g'(1)$

$$g'(a) = \frac{1}{f'(g(a))}$$

$$g'(1) = \frac{1}{f'(g(1))}$$

$f'(x) = 1 + 2x + e^x$

$$g'(1) = \frac{1}{f'(0)}$$

$$g'(1) = \frac{1}{1+2(0)+e^0}$$

$$= \boxed{\frac{1}{2}}$$

$g(1)$?

If $y = g(1)$,

then $1 = f(y)$

Solve

$1 = y + y^2 + e^y$

$y = 0$ by inspection

$g(1) = 0$

The function $f(x) = \tan x$ is one-to-one on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $g = f^{-1}$, find $g'(1)$.

$$g'(1) = \frac{1}{f'(g(1))}$$

if $g(1) = y$

then $1 = f(y)$

$1 = \tan y$

$y = \frac{\pi}{4}$ (only answer in domain)

$g(1) = \frac{\pi}{4}$

$f'(x) = \sec^2 x$

$$g'(1) = \frac{1}{f'\left(\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sec^2 \frac{\pi}{4}}$$

$$= \cos^2 \frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \boxed{\frac{1}{2}}$$