

4.2-Inverse Functions and Their Derivatives

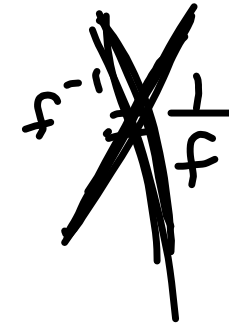
Definitions:

f is a *one-to-one* function if and only if for any y , there is a unique x such that $f(x) = y$.

To show one-to-one: Set $f(a) = f(b)$ and show ONLY solution is $a = b$

If f is one-to-one, the *inverse* of f is a function f^{-1} such that

if $y = f(x)$, then $x = f^{-1}(y)$.



If (a, b) is on the graph of $y = f(x)$, then (b, a) is on the graph of $y = f^{-1}(x)$.

If f is one-to-one and differentiable at $x = g(a)$ and $g = f^{-1}$, then

$$y = g(x) \text{ means}$$

$$x = f(y) \text{ implicit diff}$$

$$1 = f'(y) y'$$

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(g(x))}$$

so $g'(a) = \frac{1}{f'(g(a))}$

Examples:

a) Show $f(x) = \sqrt{3x-2}$ is one-to-one

b) Find the inverse of $f(x) = \sqrt{3x-2}$

a) $f(a) = f(b)$

$$(\sqrt{3a-2})^2 = (\sqrt{3b-2})^2$$

$$3a-2 = 3b-2$$

$$3a = 3b$$

$$a = b \text{ only solution,}$$

so f is one-to-one

b) $y = \sqrt{3x-2}$ switch x and y

$$x = \sqrt{3y-2} \text{ solve for } y$$

$$x^2 = 3y-2$$

$$x^2+2 = 3y$$

$$f^{-1}(x) = y = \frac{1}{3}(x^2+2)$$

Problem: NOT ONE-TO-ONE

Solution: restrict domain

$$f^{-1}(x) = \frac{1}{3}(x^2+2); x \geq 0$$

* Cannot use $x < 0$ since domain of f^{-1} must be same as range of f

Note

$$f^{-1}: D: [0, \infty)$$

$$R: [\frac{2}{3}, \infty)$$

$$f: D: [\frac{2}{3}, \infty)$$

$$R: [0, \infty)$$

since $3x-2 \geq 0$

Given $f(x) = \frac{3-x}{1-x}$, find f^{-1}

$$\frac{x}{1} = \frac{3-y}{1-y} \quad \text{Solve for } y$$

$$x(1-y) = 3-y$$

$$\underline{x} - \underline{xy} = \underline{3} - \underline{y}$$

$$y - xy = 3 - x$$

$$y(1-x) = 3-x$$

$$f^{-1}(x) = y = \frac{3-x}{1-x}$$

NOTE $f^{-1} = f$!

Given $g(x)$ is the inverse of $f(x) = x + x^2 + e^x$, find $g'(1)$

$$g'(a) = \frac{1}{f'(g(a))}$$

$$g'(1) = \frac{1}{f'(g(1))}$$

$$f'(x) = 1 + 2x + e^x$$

$$g'(1) = \frac{1}{f'(0)}$$

$$g'(1) = \frac{1}{1 + 2(0) + e^0}$$
$$= \boxed{\frac{1}{2}}$$

$g(1)?$

If $y = g(1)$,
then $1 = f(y)$

$$1 = y + y^2 + e^y \quad \text{Solve}$$

$y = 0$ by inspection

$$g(1) = 0$$

The function $f(x) = \tan x$ is one-to-one on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $g = f^{-1}$, find $g'(1)$.

$$g'(1) = \frac{1}{f'(g(1))}$$

if $g(1) = y$

then $1 = f(y)$

$$1 = \tan y$$

$$y = \frac{\pi}{4} \text{ (only answer in domain)}$$

$$g(1) = \frac{\pi}{4}$$

$$f'(x) = \sec^2 x$$

$$g'(1) = \frac{1}{f'\left(\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sec^2 \frac{\pi}{4}}$$

$$= \cos^2 \frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \boxed{\frac{1}{2}}$$