4.2-Inverse Functions and Their Derivatives

Definitions:

\( f \) is a one-to-one function if and only if for any \( y \), there is a unique \( x \) such that \( f(x) = y \).

To show one-to-one: Set \( f(a) = f(b) \) and show ONLY solution is \( a = b \).

If \( f \) is one-to-one, the inverse of \( f \) is a function \( f^{-1} \) such that

if \( y = f^{-1}(x) \), then \( x = f(y) \).

If \( (a, b) \) is on the graph of \( y = f(x) \), then \( (b, a) \) is on the graph of \( y = f^{-1}(x) \).
If $f$ is one-to-one and differentiable at $x = g(a)$ and $g = f^{-1}$, then

$$y = g(x) \text{ means } x = f(y) \text{ implicit diff }$$
$$1 = f'(y) \gamma'$$
$$\gamma' = \frac{1}{f'(y)} = \frac{1}{f'(g(x))}$$

so

$$g'(a) = \frac{1}{f'(g(a))}$$

Examples:

a) Show $f(x) = \sqrt{3x-2}$ is one-to-one.

b) Find the inverse of $f(x) = \sqrt{3x-2}$.

a) $f(a) = f(b)$

$$\left(\sqrt{3a-2}\right)^2 = \left(\sqrt{3b-2}\right)^2$$

$3a-2 = 3b-2$

$3a = 3b$

$a = b$ only solution,

so $f$ is one-to-one.

b) $y = \sqrt{3x-2}$

Switch $x$ and $y$

$x = \sqrt{3y-2}$

Solve for $y$

$x^2 = 3y-2$

$x^2 + 2 = 3y$

$x = \sqrt{\frac{1}{3} (x^2 + 2)}$

$\frac{\text{Problem: NOT}}{\text{Solution: restrict domain}}$

$\frac{\text{one-to-one}}{\text{Cannot use } x \leq 0 \text{ since domain of } f^{-1}}$

Cannot use $x \leq 0$ since domain of $f^{-1}$

must be same as range of $f$

Note

$f^{-1}: D: [0,\infty) \quad f: D [\frac{2}{3}, \infty)$

$R: [\frac{2}{3}, \infty) \quad R: [0,\infty)$
Given \( f(x) = \frac{3 - x}{1 - x} \), find \( f^{-1} \)

\[
\frac{x}{1} \times \frac{3 - y}{1 - y} = x(1 - y) = 3 - y
\]

\[
x - xy = 3 - y
\]

\[
y - xy = 3 - x
\]

\[
y(1 - x) = 3 - x
\]

\[
\boxed{f^{-1}(x) = y = \frac{3 - x}{1 - x}}
\]

**NOTE** \( f^{-1} = f \)!
Given \( g(x) \) is the inverse of \( f(x) = x + x^2 + e^x \), find \( g'(1) \)

\[
\begin{align*}
g'(1) &= \frac{1}{f'(g(1))} \\
g'(1) &= \frac{1}{f'(0)} \\
g'(1) &= \frac{1}{1 + 2x + e^x} \\
&= \frac{1}{2}
\end{align*}
\]

If \( y = g(1) \), then \( 1 = f(y) \) \( \text{Solve} \)

\[
l = y + y^2 + e^y
\]

\[
y = 0 \quad \text{by inspection}
\]

\[
g(1) = 0
\]
The function $f(x) = \tan x$ is one-to-one on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $g = f^{-1}$, find $g'(1)$.

$$g'(1) = \frac{1}{f'(g(1))}$$

$$f'(x) = \sec^2 x \quad g'(1) = \frac{1}{f'(\frac{\pi}{4})}$$

$$= \frac{1}{\sec^2 \frac{\pi}{4}}$$

$$= \cos^2 \frac{\pi}{4} = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

if $g(1) = \gamma$

then $1 = f(\gamma)$

$1 = \tan \gamma$

$\gamma = \frac{\pi}{4}$ (only answer in domain)

$g(1) = \frac{\pi}{4}$