4.5-Exponential Growth and Decay

A solution to the differential equation $y' = ky$ is: $y = Ce^{kt}$ (C, k constants)

Show :

$$(Ce^{kt})' \stackrel{?}{=} k(Ce^{kt})$$

$$Ce^{kt} \cdot k = kCe^{kt} \checkmark$$

Can show (in MATH 308) that $y = Ce^{kt}$ is the only solution to $y' = ky$

Exponential Growth/Decay:

Idea: The rate of change in a quantity is proportional to the amount present.

Mathematically: Let y = amount of quantity

$$\frac{y'}{y} = k \quad \text{OR} \quad y' = ky \quad \left(\text{Solution is } y = Ce^{kt}\right)$$

Goal: Use given information to find C and k.

Examples:

exponential decay

$t=0$

Aggigium is a radioactive substance with half-life of 105 days. If there are 2012g of Aggigium initially, how much remains after t days? How much remains after 200 days? When will there be only 100g left?

$y = Ce^{kt}$

t = time (days)

y = # grams of Aggigium

$2012 = Ce^{k \cdot 0}$ if $t=0$ $y=2012$

$2012 = C$ (in general $C = y_0$)

$y = 2012e^{kt}$ if $t=105$, $y=1006$ (half of starting value)

$1006 = 2012e^{k(105)}$

$\ln \frac{1}{2} = \cancel{\ln e}^{105k}$

$\ln\left(\frac{1}{2}\right) = 105k$

$k = \frac{1}{105} \cdot \ln\left(\frac{1}{2}\right)$

a) $\boxed{y = 2012e^{\frac{1}{105}\ln\left(\frac{1}{2}\right)t}}$

b) $t=200$ $\boxed{y = 2012e^{\frac{1}{105}\ln\left(\frac{1}{2}\right) \cdot 200} \text{ g}}$

: approx on calc

c) $y=100$

$100 = 2012e^{\frac{1}{105}\ln\left(\frac{1}{2}\right)t}$

$\frac{100}{2012} = e^{\frac{1}{105}\ln\left(\frac{1}{2}\right)t}$

$\ln\frac{100}{2012} = \frac{1}{105}\ln\left(\frac{1}{2}\right)t$

$t = \boxed{105 \frac{\ln\left(\frac{100}{2012}\right)}{\ln\left(\frac{1}{2}\right)} \text{ days}}$

k $\quad\quad$ $t=0 \quad y=6$

According to UN data, the world population at the beginning of 2000 was 6 billion and growing at a rate of 1.6% per year. Assuming an exponential growth model, estimate the world population at the beginning of 2015.

$$y = Ce^{kt}$$

$t =$ time (years since 2000)

$y =$ population (billions)

$$6 = Ce^{k(0)}$$

$$C = 6$$

$$y = 6e^{kt}$$

Given $k = .016$

$$y = 6e^{.016t}$$

in 2015 $t = 15$

$$y = 6e^{(.016)(15)}$$

$$\approx \boxed{7.627 \text{ billion people}}$$

(Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference in temperature between the object and it surroundings.) A liquid with an initial temperature of 95°C is enclosed in a metal container that is held at a constant temperature of 25°C. If the liquid cools to 50°C after 30 minutes, what will the temperature be after 1 hour?

Let t = time (minutes)

y = temp of liquid (°C)

$y' = k(y-25)$

$u' = ku$ Let $u = y-25$

$u = Ce^{kt}$ $u' = y'$

$y - 25 = Ce^{kt}$

$y = 25 + Ce^{kt}$ if $t=0, y=95$

$95 = 25 + Ce^{k \cdot 0}$

$95 = 25 + C$

$C = 70$ $C \neq y_0$!

$y = 25 + 70e^{kt}$ if $t=30$, $y=50$

$50 = 25 + 70e^{k(30)}$

$25 = 70e^{30k}$

$\ln\frac{25}{70} = \cancel{\ln e}^{30k}$

$k = \frac{1}{30}\ln\left(\frac{25}{70}\right)$

$t=60$ $y = 25 + 70e^{\frac{1}{30}\ln\left(\frac{25}{70}\right)t}$

$y = 25 + 70e^{\frac{1}{\cancel{30}}\ln\left(\frac{25}{70}\right)\cdot \cancel{60}}$ (2)

$= 25 + 70e^{2\ln\left(\frac{25}{70}\right)}$

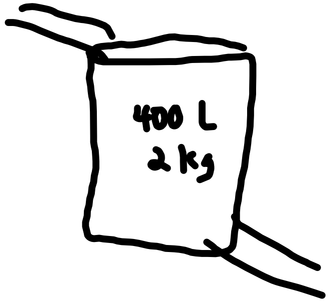
$= 25 + 70\cancel{e}^{\ln\left(\frac{25}{70}\right)^2}$

$= 25 + \cancel{70}\left(\frac{25^2}{70^{\cancel{2}}}\right)$

$= 25 + \frac{25 \cdot \cancel{25}^{5}}{\cancel{70}_{14}}$

$= 25 + \frac{125}{14} = \boxed{33\frac{13}{14}°C}$

A tank initially contains 2kg of salt dissolved in 400 liters of water. Pure water enters the tank at a rate of 20 L/min and the mixed solution is drained from the tank at the same rate. Find the amount of salt in the tank after t minutes.



$$\frac{y \text{ kg}}{\underset{20}{\cancel{400}} \text{ L}} \cdot \frac{\overset{1}{\cancel{20}} \text{ L}}{\text{min}}$$

concentration

Let t = time (min)
y = # kg salt

$$\frac{dy}{dt} = \underset{(\text{pure } H_2O)}{\text{rate in}} - \text{rate out}$$

$$= 0 - \frac{y}{20}$$

$$y' = \frac{-1}{20} y$$

$$y = Ce^{-\frac{1}{20}t}$$

if $t=0, y=2$

$$2 = C\cancel{e^{-\frac{1}{20}(0)}}$$

$C=2$

$$\boxed{y = 2e^{-\frac{1}{20}t}}$$