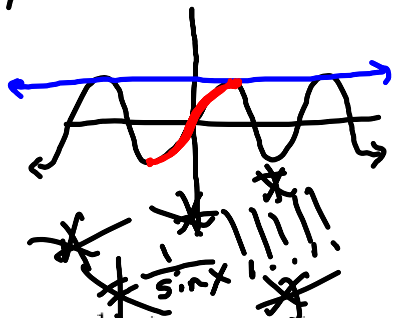
4.6-Inverse Trig Functions and their Derivatives

sin, cos tan and one-to-one functions:

$y = \sin x$



Restrict domain:

$y = \sin x$; $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is one-to-one

$y = \sin^{-1} x$ (or $\arcsin x$) if and only if $x = \sin y$ and $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ Domain $x \in [-1, 1]$

$y = \cos^{-1} x$ if and only if $x = \cos y$ and $y \in [0, \pi]$ Domain: $x \in [-1, 1]$

$y = \tan^{-1} x$ if and only if $x = \tan y$ and $y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ Domain: $(-\infty, \infty)$

Derivative of $\sin^{-1}$:

$y = \sin^{-1} x$ means

$x = \sin y$

$1 = (\cos y)\, y'$

$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

$\left(= \frac{1}{\cos(\sin^{-1} x)} \right)$

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}}$$

$\cos(\sin^{-1} x)$?

$\sin y = \frac{x}{1}$ $\frac{\text{Opp}}{\text{Hyp}}$

$\sqrt{1-x^2}$ by Pythagoras

$\cos y = \frac{\sqrt{1-x^2}}{1}$ $\frac{\text{Adj}}{\text{Hyp}}$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$$

Examples:

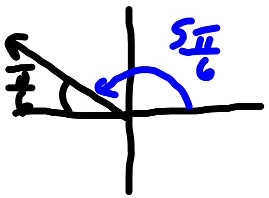
Find the exact values of each of the following:

a) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$

$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

Since $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Answer in Quad II



b) $\arcsin(-1) = -\frac{\pi}{2}$

Since $\sin -\frac{\pi}{2} = -1$

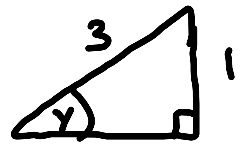
NOTE: $\frac{3\pi}{2}$ is NOT in the range of $\arcsin x$!

Compute $\cos(\underbrace{\sin^{-1}\left(\frac{1}{3}\right)}_{y})$

$$\boxed{\cos y = \frac{2\sqrt{2}}{3}}$$

$y = \sin^{-1}\frac{1}{3}$ means

$\sin y = \frac{1}{3}\ \frac{O}{H}$



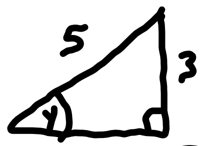
$2\sqrt{2}\ \left(\sqrt{3^2 - 1^2}\right)$

Compute $\cancel{\sin}(2\,\cancel{\arcsin}\left(\frac{3}{5}\right)) = 2\left(\frac{3}{5}\right) = \cancel{\frac{6}{5}}$ Cannot be larger than 1

$\sin(2\,\underbrace{\arcsin\left(\frac{3}{5}\right)}_{y})$

$y = \arcsin\frac{3}{5}$ means

$\sin y = \frac{3}{5}$



$4 = \sqrt{5^2 - 3^2}$

$$\sin(2y) = 2\sin y\cos y$$
$$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$$
$$= \boxed{\frac{24}{25}}$$

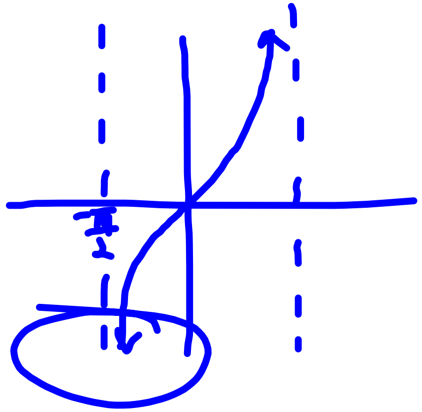
Compute $\lim_{x\to\infty} \tan^{-1}\left(\frac{1-x^3}{2x^2-x}\right)$

$$\tan^{-1}\left(\lim_{x\to\infty}\left(\frac{1-x^3}{2x^2-x}\right)\right)$$

$$\text{" } \tan^{-1}(-\infty) \text{ "} = -\frac{\pi}{2}$$

Since $\lim_{x\to -\frac{\pi}{2}^+} \tan x = -\infty$

$$\lim_{x\to\infty} \frac{\cancel{x^2}^{\,x}\left(\cancel{\frac{1}{x^3}}^{\,0}-1\right)}{\cancel{x^2}\left(2-\cancel{\frac{1}{x}}^{\,0}\right)} = \lim_{x\to\infty} -\frac{1}{2}x = -\infty$$



Product Rule

Find the derivative of $f(x) = x\ \arctan(2x) - \frac{1}{4}\ln(1+4x^2)$

$$f'(x) = (1)\arctan(2x) + x \cdot \frac{1}{1+(2x)^2} \cdot 2 - \frac{1}{\cancel{4}} \cdot \frac{1}{1+4x^2} \cdot \overset{2}{\cancel{8}}x$$

Simplify

$$= \arctan(2x) + \frac{2x}{1+4x^2} - \frac{2x}{1+4x^2}$$

$$= \boxed{\arctan(2x)}$$