

4.8 L'Hospital's Rule

Goal: Given a limit of indeterminate form ($0/0$, ∞/∞ , etc) with differentiable functions, find the limit.

L'Hospital's Rule: If f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a), and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ or $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Examples:

Find the exact values of each of the following limits:

$$\lim_{x \rightarrow -1} \frac{x^2 + 6x + 5}{x^2 - 3x - 4} \frac{0}{0} = \lim_{x \rightarrow -1} \frac{2x + 6}{2x - 3} = \frac{-2 + 6}{-2 - 3} = \boxed{\frac{-4}{5}}$$

NOTE Using factor and cancel

$$\lim_{x \rightarrow -1} \frac{(x+5)(\cancel{x+1})}{(x-4)(\cancel{x+1})} = \frac{-4}{5}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{2x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{4x} = \lim_{x \rightarrow 0} \frac{\cos x}{4} = \boxed{\frac{1}{4}}$$

The image shows a handwritten mathematical derivation. The first term is $\lim_{x \rightarrow 0} \frac{1 - \cos x}{2x^2}$ with blue circles around the numerator and denominator. This is followed by an equals sign and $\lim_{x \rightarrow 0} \frac{\sin x}{4x}$. A red circle is drawn around the fraction $\frac{\sin x}{4x}$, with the word "OK" written above it and an arrow pointing to the fraction. Below this is another equals sign and $\lim_{x \rightarrow 0} \frac{\cos x}{4}$. The final result is $\frac{1}{4}$ enclosed in a hand-drawn box.

L'Hospital's Rule ONLY applies to $\frac{f(x)}{g(x)}$

$$\lim_{x \rightarrow 0} x \ln|x|$$

$$= \lim_{x \rightarrow 0} \frac{\ln|x|}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

$$= \lim_{x \rightarrow 0} -x$$

$$= \boxed{0}$$

NOTE: Must be indeterminate to apply L'Hospital's Rule!

$$\underline{\text{Ex}} \quad \lim_{x \rightarrow 3} \frac{x^2 - 9}{x + 3} = \frac{0}{6} = 0$$

$$= \lim_{x \rightarrow 3} \frac{2x}{1} = 6$$

0 (approaching! Not equal!)

$$\lim_{x \rightarrow \infty} (1 + e^{2x})^{\frac{1}{x}}$$

Step 1:

*
*
*
e
*
*
*
1

$$\lim_{x \rightarrow \infty} \ln((1 + e^{2x})^{\frac{1}{x}}) \neq$$

Look at $\lim_{x \rightarrow \infty} \ln((1 + e^{2x})^{\frac{1}{x}})$

$$= \lim_{x \rightarrow \infty} \frac{1}{x} \ln(1 + e^{2x})$$

0 · ∞

$$= \lim_{x \rightarrow \infty} \frac{\ln(1 + e^{2x})}{x}$$

∞ / ∞

Now apply L'Hospital's Rule

$$= \lim_{x \rightarrow \infty} \frac{2e^{2x}}{1 + e^{2x}} = \lim_{x \rightarrow \infty} \frac{2e^{2x}}{1 + e^{2x}}$$

∞ / ∞

$$= \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2e^{2x}} = 2$$

Captions:
1) don't terms
2) write answer
3) L'Hospital's Rule

$$\text{So } \lim_{x \rightarrow \infty} (1 + e^{2x})^{\frac{1}{x}} = \boxed{e^2}$$



$$e^{\ln x} = x$$

End Start

Recall the formula for computing compound interest (4.3): $A = P \left(1 + \frac{r}{m}\right)^{mt}$. Find $\lim_{m \rightarrow \infty} A$.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} P \left(1 + \frac{r}{n}\right)^{nt} &= P \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{nt} \\
 &= P e^{\lim_{n \rightarrow \infty} \ln \left(1 + \frac{r}{n}\right)^{nt}} \\
 &= P e^{\lim_{n \rightarrow \infty} nt \ln \left(1 + \frac{r}{n}\right)} \\
 &= P e^t \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{r}{n}\right)}{\frac{1}{n}} \quad \ln \left(\frac{n+r}{n}\right) \\
 &= P e^t \left(\lim_{n \rightarrow \infty} \frac{1}{1 + \frac{r}{n}} \cdot \frac{-\frac{r}{n^2}}{\frac{1}{n^2}} \right) \\
 &= P e^{t \cdot r} = \boxed{P e^{rt}} \quad \text{Continuous Compound Interest}
 \end{aligned}$$