4.8 L'Hospital's Rule

Goal: Given a limit of indeterminate form (0/0, ∞/∞, etc) with differentiable functions, find the limit.

L'Hospital's Rule: If f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a), and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Examples:

Find the exact values of each of the following limits:

$$\lim_{x\to -1} \frac{x^2+6x+5}{x^2-3x-4} \quad \frac{0}{0} = \lim_{x\to -1} \frac{2x+6}{2x-3} = \frac{-2+6}{-2-3} = \boxed{-\frac{4}{5}}$$

NOTE Using factor and cancel

$$\lim_{x\to -1} \frac{(x+5)\cancel{(x+1)}}{(x-4)\cancel{(x+1)}} = -\frac{4}{5}$$

$$\lim_{x\to 0} \frac{1-\cos x}{2x^2} \quad \frac{0}{0} = \lim_{x\to 0} \frac{\sin x}{4x} \xrightarrow{\text{ok}} 1$$

$$= \lim_{x\to 0} \frac{\cos x}{4} = \boxed{\frac{1}{4}}$$

L'Hospital's Rule ONLY applies to $\frac{f(x)}{g(x)}$

$0 \cdot -\infty$

$$\lim_{x\to 0} x \ln|x| = \lim_{x\to 0} \frac{\ln|x|}{\frac{1}{x}} \quad \frac{-\infty}{\pm\infty}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \quad -x^2$$

$$= \lim_{x\to 0} -x$$

$$= \boxed{0}$$

NOTE: Must be indeterminate to apply L'Hospital's Rule!

Ex $\lim_{x\to 3} \frac{x^2-9}{x+3} \quad \frac{0}{6} = 0$

" $= \lim_{x\to 3} \frac{2x}{1} = 6$ "

$$\lim_{x\to\infty}(1+e^{2x})^{\frac{1}{x}}$$

∞

0 (approaching! Not equal!)

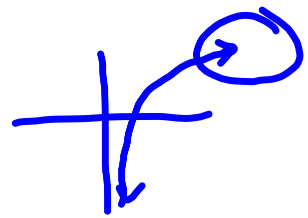
Step 1:

$$e \quad \lim_{x\to\infty} \ln\left((1+e^{2x})^{\frac{1}{x}}\right) \neq$$

Look at $\lim_{x\to\infty} \ln\left((1+e^{2x})^{\frac{1}{x}}\right)$

$$= \lim_{x\to\infty} \frac{1}{x}\ln(1+e^{2x})$$

$0 \cdot \infty$



$$= \lim_{x\to\infty} \frac{\ln(1+e^{2x})}{x} \quad \frac{\infty}{\infty}$$

Now apply L'Hospital's Rule

$$= \lim_{x\to\infty} \frac{\frac{2e^{2x}}{1+e^{2x}}}{1} = \lim_{x\to\infty} \frac{2e^{2x}}{1+e^{2x}} \quad \frac{\infty}{\infty}$$

Options:
1) dom terms
2) write answer
3) L'Hospital again

$$= \lim_{x\to\infty} \frac{\cancel{4}\,2\,\cancel{e^{2x}}}{\cancel{2}\,\cancel{e^{2x}}} = 2$$

$$\text{So } \lim_{x\to\infty}(1+e^{2x})^{\frac{1}{x}} = \boxed{e^2}$$

$$e^{\ln x} = x$$

End Start

Recall the formula for computing compound interest (4.3): $A = P\left(1+\dfrac{r}{m}\right)^{mt}$. Find $\lim\limits_{m\to\infty} A$.

$$\lim_{m\to\infty} P(\underbrace{1+\tfrac{r}{m}}_{1})^{mt\,\infty} = P\lim_{m\to\infty}\left(1+\tfrac{r}{m}\right)^{mt}$$

$$= Pe^{\lim\limits_{m\to\infty}\ln\left(\left(1+\frac{r}{m}\right)^{mt}\right)}$$

$$= Pe^{\lim\limits_{m\to\infty} mt\ln\left(1+\frac{r}{m}\right)}$$

$$\infty \cdot 0$$

$$\ln\left(\frac{m+r}{m}\right)$$

$$= Pe^{t\lim\limits_{m\to\infty}\frac{\ln\left(1+\frac{r}{m}\right)}{\frac{1}{m}}} \quad \frac{0}{0}$$

$$= Pe^{t\left(\lim\limits_{m\to\infty}\frac{\frac{1}{1+\frac{r}{m}\to 0}\cdot\frac{-r}{\cancel{m^2}}}{\frac{-1}{\cancel{m^2}}}\right)}$$

$$= Pe^{t\cdot r} = \boxed{Pe^{rt}}$$

Continuous Compound Interest