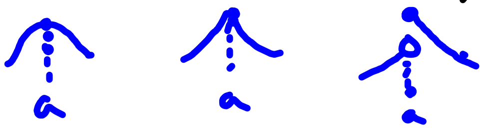
5.2-Maxima and Minima

Definitions: (local max)

f has a *relative maximum* at $x = a$ if and only if $f(a) > f(x)$ for all x "near" a.



f has a *relative minimum* at $x = a$ if and only if $f(a) < f(x)$ for all x "near" a.

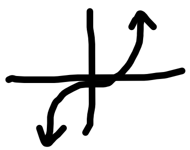
Fermat's Theorem: If f has a relative maximum or relative minimum at $x = a$ and f is differentiable at $x = a$, then $f'(a) = 0$

More definitions:

f has a critical value at $x = a$ if and only if $f'(a) = 0$ or $f'(a)$ DNE

NOTE: crit val does not guarantee max or min

$f(x) = x^3$



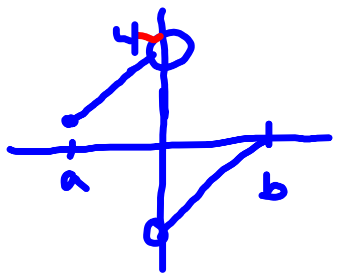
$f'(0) = 0$ but not max or min

f has an *absolute maximum* at $x = a$ if and only if $f(a) \geq f(x)$ for all x

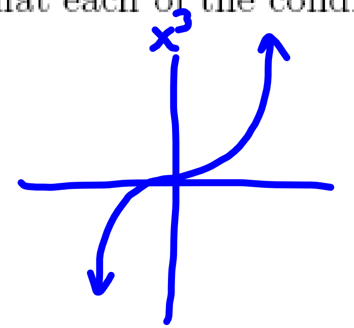
f has an *absolute minimum* at $x = a$ if and only if $f(a) \leq f(x)$ for all x

Extreme Value Theorem If f is continuous on a closed, bounded interval, then f attains its absolute max and absolute min on that interval.

Graphical examples to show that each of the conditions must hold to guarantee the conclusion:



No abs max
No abs min



No abs max on $(-\infty, \infty)$
No abs min on $(-\infty, \infty)$

Where can abs max/min occur?

1) crit value ($f' = 0$ or f' DNE)

2) endpoints

Examples:

cts

Find the absolute maximum and absolute minimum of $f(x) = 2x^3 - 15x^2 + 36x + 7$ on the interval $[1, 5]$ closed bdd

$f'(x) = 6x^2 - 30x + 36 = 0$

$6(x^2 - 5x + 6) = 0$

$6(x-2)(x-3) = 0$

$x = 2 \quad x = 3$

$f(1) = 2 - 15 + 36 + 7 = 30$

$f(2) = 16 - 60 + 72 + 7 = 35$

$f(3) = 54 - 135 + 108 + 7 = 34$

$f(5) = 250 - 375 + 180 + 7 = 62$

Abs max is 62 when $x=5$
Abs min is 30 when $x=1$

* Note: correction from 9:35 class! *

cts on interval

Find the absolute maximum and absolute minimum of $f(x) = 2\sec x - \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$. closed bdd

$$f'(x) = 2\sec x \tan x - \sec^2 x = 0$$

$$\frac{2\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} = 0$$

$$\frac{2\sin x - 1}{\cos^2 x} = 0$$

f' DNE when $x = \frac{\pi}{2}$ (not in interval)

$$2\sin x - 1 = 0$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \cancel{\frac{5\pi}{6}}$$

$$f(0) = 2\sec 0 - \tan 0 = 2$$

$$f\left(\frac{\pi}{6}\right) = 2\sec\frac{\pi}{6} - \tan\frac{\pi}{6} = \frac{4}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$f\left(\frac{\pi}{4}\right) = 2\sec\frac{\pi}{4} - \tan\frac{\pi}{4} = 2\sqrt{2} - 1 \approx 1.8$$

$$\boxed{\begin{array}{l}\text{Abs max of } 2 \text{ when } x = 0 \\ \text{Abs min of } \sqrt{3} \text{ when } x = \frac{\pi}{6}\end{array}}$$

cts

Find the abslute maximum and absolute minimum of $f(x) = \dfrac{\ln x}{x}$ on the interval $(0, \infty)$

Not closed/bdd

Find c.v.: $f'(x) = \dfrac{x \cdot \frac{1}{x} - \ln x(1)}{x^2} = 0$

f' DNE at $x=0$ not in interval

$1 - \ln x = 0$

$e^{1} = \cancel{e^{\ln}}\, x$

$x = e$

Rel max or min?

f: $\nearrow$ $\searrow$

f': $+ + + + + +$ $\quad$ $- - - - - - -$

e

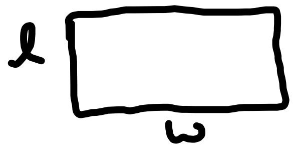
Test $x=1$: $f'(1)$: $\dfrac{1 - \ln 1}{1^2} > 0$

Test $x = e^2$: $f'(e^2)$: $\dfrac{1 - \cancel{\ln(e^2)}\,2}{e^4}$ $\dfrac{-}{+}$

Rel max at $x = e$

ONLY critical value, so $\boxed{\text{abs max at } x=e \text{ of } \frac{\ln e}{e} = \frac{1}{e}. \quad \text{No abs min}}$

Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.



Goal: Max $A = \ell w$

Restriction: $P = 2\ell + 2w = 100$ solve for w

$2w = 100 - 2\ell$

$w = 50 - \ell$

Goal: Max $A = \ell(50 - \ell)$

$A = 50\ell - \ell^2$; $0 \le \ell \le 50$

c.v. $A' = 50 - 2\ell = 0$

$2\ell = 50$

$\ell = 25$

$A(0) = 0$

$A(25) = 25(50-25) = 625$ Max

$A(50) = 0$

$\ell = 25$ so $w = 50 - 25 = 25$

$\boxed{25 \times 25 \text{ m}}$