5.2-Maxima and Minima

Definitions: 

- A function $f$ has a \textit{relative maximum} at $x = a$ if and only if $f(a) \geq f(x)$ for all $x$ "near" $a$.

- A function $f$ has a \textit{relative minimum} at $x = a$ if and only if $f(a) \leq f(x)$ for all $x$ "near" $a$.

\textbf{Fermat’s Theorem:} If $f$ has a relative maximum or relative minimum at $x = a$ and $f$ is differentiable at $x = a$, then $f'(a) = 0$.

More definitions:

- A function $f$ has a \textit{critical value} at $x = a$ if and only if $f'(a) = 0$ or $f'(a)$ DNE

\textbf{Note:} A critical value does not guarantee a max or min.

- Consider $f(x) = x^3$ where $f'(0) = 0$ but not max or min.
\( f \) has an absolute maximum at \( x = a \) if and only if \( f(a) \geq f(x) \) for all \( x \)

\( f \) has an absolute minimum at \( x = a \) if and only if \( f(a) \leq f(x) \) for all \( x \)

**Extreme Value Theorem** If \( f \) is continuous on a closed, bounded interval, then \( f \) attains its absolute max and absolute min on that interval.

Graphical examples to show that each of the conditions must hold to guarantee the conclusion:

- No abs max
  - No abs min

- No abs max on \((-\infty, \infty)\)
  - No abs min on \((-\infty, \infty)\)

Where can abs max/min occur?
1) crit value (\( f' = 0 \))
2) endpoints
Examples:

Find the absolute maximum and absolute minimum of \( f(x) = 2x^3 - 15x^2 + 36x + 7 \) on the interval \([1, 5]\).  

\[
f'(x) = 6x^2 - 30x + 36 = 0
\]

\[
6(x^2 - 5x + 6) = 0
\]

\[
6(x - 2)(x - 3) = 0
\]

\[
x = 2 \quad x = 3
\]

\[
f(1) = 2 - 15 + 36 + 7 = 30
\]

\[
f(2) = 16 - 60 + 72 + 7 = 35
\]

\[
f(3) = 54 - 135 + 108 + 7 = 34
\]

\[
f(5) = 250 - 375 + 180 + 7 = 62
\]

Abs max is 62 when \( x = 5 \)
Abs min is 30 when \( x = 1 \)
Find the absolute maximum and absolute minimum of \( f(x) = 2\sec x - \tan x \) on the interval \([0, \pi/4]\). The critical points occur where the derivative is zero:

\[
\frac{2\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} = 0
\]

Solving for \( x \), we get:

\[
\frac{2\sin x - 1}{\cos^2 x} = 0
\]

\[
2\sin x - 1 = 0
\]

\[
\sin x = \frac{1}{2}
\]

\[
x = \frac{\pi}{6}
\]

\[
Abs \ max \ of \ 2 \ when \ x = 0
\]

\[
Abs \ min \ of \ \sqrt{3} \ when \ x = \pi/6
\]

\[
f(0) = 2\sec 0 - \tan 0 = 2
\]

\[
f(\pi/6) = 2\sqrt{3} - \frac{\sqrt{3}}{3} = \frac{6}{3} - \frac{1}{3} = \frac{5}{3} = \sqrt{3}
\]

\[
f(\pi/4) = 2\sec \frac{\pi}{4} - \tan \frac{\pi}{4} = 2\cdot2\sqrt{2} - 1 = 4\sqrt{2} - 1 \approx 1.8
\]
Find the absolute maximum and absolute minimum of \( f(x) = \frac{\ln x}{x} \) on the interval \((0, \infty)\).

Find c.v.:
\[
f'(x) = \frac{x \cdot \frac{1}{x} - \ln x (1)}{x^2} = 0
\]

1. \( f' \) DNE at \( x = 1 \).
2. \( f' \) not in interval.

Rel max or min?

\[
\begin{array}{c}
\text{Test} x = 1 \\
\text{Test} x = e^2 \\
\text{Test} x = e
\end{array}
\]

\[
\begin{array}{cc}
f'(1) & f'(e^2) \\
1 - \ln 1 & 1 - (2e^2) \\
0 & -
\end{array}
\]

Rel max at \( x = e \)

ONLY critical value, so abs max at \( x = e \) of \( \frac{\ln e}{e} = \frac{1}{e} \).

No abs min.
Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

\[ l \quad w \]

Goal: Max \( A = lw \)

Restriction: \( P = 2l + 2w = 100 \) \( \text{solve for } w \)

\[ 2w = 100 - 2l \]
\[ w = 50 - l \]

Goal: Max \( A = l(50-l) \)
\[
A = 50l - l^2 ; \quad 0 \leq l \leq 50
\]

C.V. \( A' = 50 - 2l = 0 \)
\[ 2l = 50 \]
\[ l = 25 \]

\( A(0) = 0 \)
\( A(25) = 25(50-25) = 625 \) \( \text{Max} \)
\( A(50) = 0 \)

\[ l = 25 \text{ so } w = 50 - 25 = 25 \]

\[ 25 \times 25 \text{ m} \]