5.3-Derivatives and the Shapes of Curves

Mean Value Theorem: If f cts $[a,b]$ and diff (a,b) then there is a $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Recall 5.1: What f' and f'' say about f:

if $f' > 0$; f is inc

$f' < 0$; f is dec

if $f'' > 0$; f is conc up and f' is inc

$f'' < 0$; f is conc down and f' is dec

Second Derivative Test: if f has a crit val at $x=a$, and:

$\cup$ 1) $f''(a) > 0$, then f has a rel min at $x=a$

$\cap$ 2) $f''(a) < 0$, then f has a rel max at $x=a$

Examples:

$f' > 0$ $\quad$ $f' < 0$

Determine where the function $f(x) = x^3 - 3x^2 + 1$ is increasing and decreasing, concave up, and concave down.

Find crit vals: $\quad f'(x) = 3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0 \quad x = 2$

f: ↗ ↘ ↗

f': $++++$ $\quad$ $----$ $\quad$ $++++$

$0 \quad 2$

Test $x=-1$: $3(-1)(-1-2)$, $- \cdot -$

Test $x=1$: $3(1)(1-2)$, $+ \cdot -$

Test $x=3$: $3(3)(3-2)$, $+ \cdot +$

$$\boxed{\begin{array}{l} f \text{ inc } (-\infty, 0) \cup (2, \infty) \\ f \text{ dec } (0, 2) \end{array}}$$

rel max at $x = 0$

rel min at $x = 2$

Repeat using f'': $\quad f''(x) = 6x - 6 = 0$

$x = 1$

f: ∩ ∪

f': ↘ ↗

f'': $------$ $\quad$ $++++++$

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Test $x=0$: $f''(0)$, $6 \cdot 0 - 6$

Test $x=2$: $f''(2)$, $6 \cdot 2 - 6$

$$\boxed{\begin{array}{l} f \text{ conc up } (1, \infty) \\ f \text{ conc down } (-\infty, 1) \end{array}}$$

infl pt at $x = 1$

$f' > 0 \quad f' < 0 \quad f'' > 0 \quad f'' < 0$

Determine where the function $f(x) = x^2e^{-2x}$ is increasing, decreasing, concave up, and concave down.

$$f'(x) = 2xe^{-2x} + x^2(-2e^{-2x}) = 0$$

$$2x\underbrace{e^{-2x}}_{\neq 0}(1-x) = 0$$

$x = 0 \qquad x = 1$

$f \quad \searrow \quad \nearrow \quad \searrow$

$f' \quad - - \quad + + + \quad - - -$

$0 \qquad 1$

Test $x=-1$	Test $x=\frac{1}{2}$	Test $x=2$
$f'(-1)$	$f'(\frac{1}{2})$	$f'(2)$
$- + +$	$+ + +$	$+ + -$

$$\boxed{\begin{array}{l} f \text{ inc } (0,1) \\ f \text{ dec } (-\infty,0) \cup (1,\infty) \end{array}}$$

$$f'(x) = e^{-2x}(2x - 2x^2)$$

$$f''(x) = -2e^{-2x}(2x-2x^2) + e^{-2x}(2-4x) = 0$$

$$e^{-2x}(-4x + 4x^2 + 2 - 4x) = 0$$

$$e^{-2x}(4x^2 - 8x + 2) = 0$$

$$2\underbrace{e^{-2x}}_{\neq 0}(2x^2 - 4x + 1) = 0$$

Quad Form

$$x = \frac{4 \pm \sqrt{16 - 4(2)(1)}}{2(2)}$$

$$= \frac{4 \pm 2\sqrt{2}}{4} = \frac{2 \pm \sqrt{2}}{2}$$

$f \quad \cup \quad \cap \quad \cup$

$f' \quad \nearrow \quad \searrow \quad \nearrow$

$f'' \quad + \quad - \quad +$

$f''(0) \quad \frac{2-\sqrt{2}}{2} \quad f''(1) \quad \frac{2+\sqrt{2}}{2} \quad f''(2)$

$$\boxed{\begin{array}{l} \text{conc up } \left(-\infty, \frac{2-\sqrt{2}}{2}\right) \cup \left(\frac{2+\sqrt{2}}{2}, \infty\right) \\ \text{conc down } \left(\frac{2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}\right) \end{array}}$$

2nd der

Find the inflection points of $f(x) = -x^2 \cos x + 6 \cos x + 4x \sin x$, $x \in [-\pi, \pi]$.

$$f'(x) = -2x\cos x + x^2 \sin x - 6 \sin x + 4 \sin x + 4x \cos x$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x$$

$$f''(x) = \cancel{2x \sin x} + x^2 \cos x + \cancel{2\cos x} - \cancel{2x\sin x} - \cancel{2\cos x}$$

$$f''(x) = x^2 \cos x = 0$$

not infl $\cancel{x = 0}$ $\therefore$ $\boxed{x = \frac{-\pi}{2},\ x = \frac{\pi}{2}}$ ONLY

f	$\cap$	$\cup$	$\cup$	$\cap$
f''	$-$	$+$	$+$	$-$
	$f''(-\pi)$	$-\frac{\pi}{2}$ $f''(\frac{-\pi}{4})$ 0	$f''(\frac{\pi}{4})$ $\frac{\pi}{2}$	$f''(\pi)$
	$+\ -$	$+\ +$	$+\ +$	$+\ -$