6.2-6.3: The Definite Integral

Definitions:

partition- P of an interval $[a,b]$ is a set of numbers $P=\{x_0,x_1,x_2,\ldots,x_n\}$ such that $a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$

$\triangle x_i$: $x_i - x_{i-1}$

$\|P\|$: (norm of partition) $=\max(\Delta x_i)$

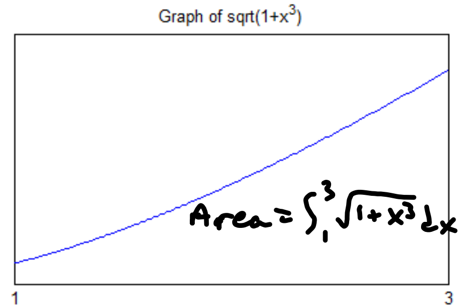
x_i^*: a (predefined) element of $[x_{i-1},x_i]$

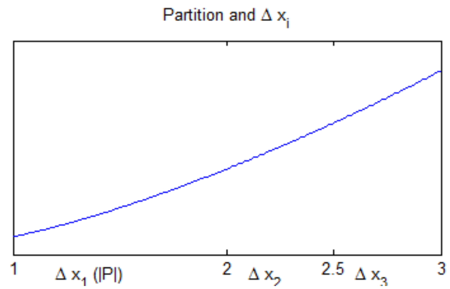
A *Riemann Sum* $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ $\left(= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + f(x_3^*)\Delta x_3 + \ldots + f(x_n^*)\Delta x_n\right)$

The **definite integral** of a function f from $x=a$ to $x=b$:

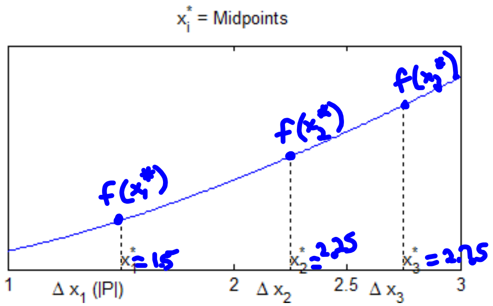
$$\int_a^b f(x)\,dx = \lim_{|P|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

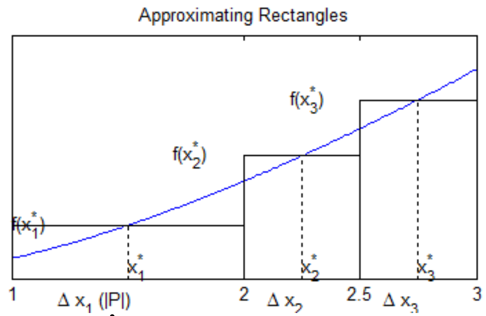
Illustrate: $\int_1^3 \sqrt{1+x^3}\,dx$ (area under graph from 1 to 3)





$P = \{1, 2, 2.5, 3\}$



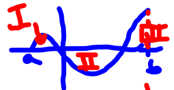


$A = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + f(x_3^*)\Delta x_3$

Approximation to $\int_1^3 f(x)\,dx$

(NOTE: If $f(x) \geq 0$ on $[a,b]$, then the definite integral is the area under the graph from $x=a$ to $x=b$).

If not, $\int_a^b f(x)\,dx$ = Area Above − Area Below



$\int_a^b f(x)\,dx = I + III - II$

Examples:

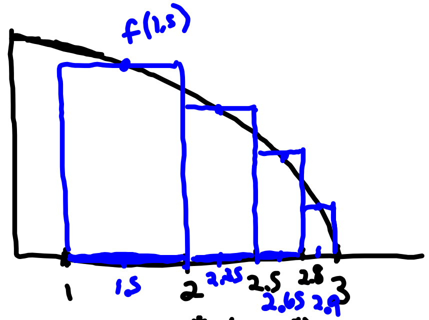
Given $f(x) = 9 - x^2$, write and compute a Riemann Sum to approximate $\int_1^3 f(x)\,dx$ using a partition of $P = \{1, 2, 2.5, 2.8, 3\}$. Let x_i^* = the midpoint of each subinterval.

$$\int_a^b f(x)\,dx = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

$$\approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

$$\int_1^3 (9-x^2)\,dx \approx \sum_{i=1}^{4} f(x_i^*)\,\Delta x_i$$

$$= (9-1.5^2)(1) + (9-2.25^2)(0.5) + (9-2.65^2)(0.3) + (9-2.9^2)(0.2)$$



i Rect	Δx_i	x_i^*	$f(x_i^*)\Delta x_i$
1	2−1 = 1	1.5	$(9-1.5^2)(1)$
2	2.5−2 = 0.5	2.25	$(9-2.25^2)(0.5)$
3	2.8−2.5 = 0.3	2.65	$(9-2.65^2)(0.3)$
4	3−2.8 = 0.2	2.9	$(9-2.9^2)(0.2)$

Equally-spaced partitions: Let n be the number of equally-spaced subintervals of $[a, b]$. Then $\Delta x_i = \frac{b-a}{n}$ NOTE: as $n \to \infty$, $\|P\| \to 0$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\frac{b-a}{n}}_{\Delta x_i} \sum_{i=1}^{n} f\Big(\underbrace{a + i\,\frac{b-a}{n}}_{x_i^*}\Big)$$

NOTE: Given if needed

Given $f(x) = 9 - x^2$, find the exact value of $\int_1^3 f(x)\,dx$ from the definition.

$a = 1$
$b = 3$
$f(x) = 9 - x^2$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\,\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(9 - \left(1 + \frac{2i}{n}\right)^2\right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(9 - \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right)\right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(8 - \frac{4i}{n} - \frac{4i^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \left(\sum_{i=1}^{n} 8 - \sum_{i=1}^{n} \frac{4i}{n} - \sum_{i=1}^{n} \frac{4i^2}{n^2}\right)$$

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$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{2}{n} \left(\sum_{i=1}^{n} 8 - \frac{4}{n}\sum_{i=1}^{n} i - \frac{4}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \left(8n - \frac{4}{n}\cdot\frac{n(n+1)}{2} - \frac{4}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right)$$

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$$= \lim_{n\to\infty} \left(16 - \frac{8(n+1)}{2n} - \frac{4(n+1)(2n+1)}{6n^2}\right) \quad \frac{4(2n^2+3n+1)}{3n^2}$$

$$= 16 - 4 - \frac{8}{3} = \boxed{\frac{28}{3}}$$

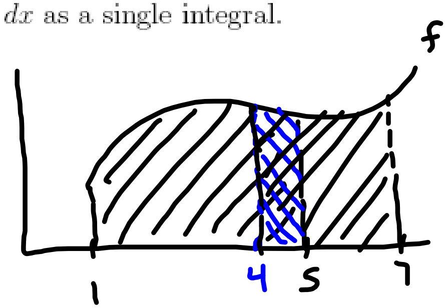
Properties of Definite Integrals: (pp383-385)
(NOTE: Some of the more useful properties for future sections are #2, 3, 5 and 8).

Examples:

Rewrite $\int_1^5 f(x)\,dx - \int_4^5 f(x)\,dx + \int_4^7 f(x)\,dx$ as a single integral.

$= \int_1^4 f(x)dx + \int_4^7 f(x)\,dx$

$= \boxed{\int_1^7 f(x)\,dx}$



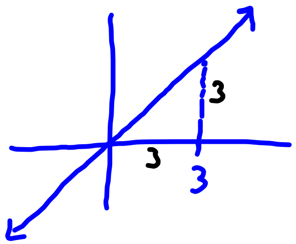
Compute $\int_0^3 (x - \sqrt{9 - x^2})\, dx$

$$= \int_0^3 x\, dx - \int_0^3 \sqrt{9-x^2}\, dx$$

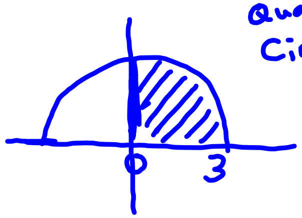
$$= \frac{1}{2}(3)(3) - \frac{1}{4}\cdot\pi\cdot 3^2$$

$$= \frac{9}{2} - \frac{9\pi}{4}$$

Triangle Area $= \int_0^3 x\, dx$



Area Quarter Circle



$y = \sqrt{9-x^2}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$ (Circle)