6.5-Substitution

Recall: Chain Rule for Derivatives: $\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$

Therefore $\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$

Problem: Recognizing when you have an integral of this form and what f and g are.

Solution: Substitute for $g(x)$, your "inner function" $u = g(x)$

Examples:

Find the area under the graph of $f(x) = \sin(Bx)$, from $x = 0$ to $x = \frac{\pi}{B}$.

$$A = \frac{1}{B}\int_0^{\pi/B} \sin(Bx)\,B\,dx$$

Let $u = Bx$

$du = B\cdot dx$

(Recall: $dy = f'(x)dx$)

Alt: $dx = \frac{du}{B}$

$$= \frac{1}{B}\int_{\cancel{0}}^{\cancel{\pi/B}} \sin(u)\,du$$

Correct:

$\frac{1}{\text{stuff}}$

$$= \frac{1}{B}\int_{x=0}^{x=\pi/B} \sin u\,du$$

$$= \frac{1}{B}\left.-\cos u\right|_{x=0}^{x=\frac{\pi}{B}}$$

$$= \frac{1}{B}\left(-\cos Bx\right)\Big|_0^{\pi/B}$$

$$= \frac{1}{B}\left(-\cos B\cdot\tfrac{\pi}{B} + \cos B\cdot 0\right)$$

$+1 \qquad +1$

$$= \frac{1}{B}(1+1) = \boxed{\frac{2}{B}}$$

Alternate: change boundaries to new axis

Let $u = Bx$ $\quad x = 0 \quad u = B\cdot 0 = 0$

$x = \frac{\pi}{B} \quad u = B\cdot\frac{\pi}{B} = \pi$

$$A = \frac{1}{B}\int_0^{\pi} \sin u\,du$$

$$= \frac{1}{B}\left(-\cos u\Big|_0^{\pi}\right)$$

$$= \frac{1}{B}\left(-\overset{-1}{\cos\pi} + \overset{1}{\cos 0}\right)$$

$$= \frac{1}{B}(2) = \boxed{\frac{2}{B}}$$

Compute $\int_0^1 x(2x^2-1)^{10}\,dx$

Let $u = 2x^2 - 1$

$du = 4x\,dx$

$dx = \frac{du}{4x}$

$x=0 \rightarrow u=-1$

$x=1 \rightarrow u=1$

$= \int \cancel{x} \cdot u^{10} \cdot \frac{du}{4\cancel{x}}$

$= \frac{1}{4}\int_{-1}^{1} u^{10}\,du$

NOTE:

$\frac{2}{4}\int_0^1 u^{10}\,du$

$= \frac{1}{4} \cdot \frac{1}{11} u^{11} \Big|_{-1}^{1}$

$= \boxed{\frac{1}{44}\left(1-(-1)\right)} = \frac{2}{44} = \boxed{\frac{1}{22}}$

NOTE:

$\int_0^1 \cancel{x}(2x^2-1)^{10}\,dx$

$\overline{\text{"}4\cancel{x}\text{"}}$

$= \frac{1}{4} \cdot \frac{1}{11}(2x^2-1)^{11}\Big|_0^1$

Compute $\frac{1}{2}\int \frac{2e^{2x}}{(e^{2x}+2)}\,dx$

Let $u = e^{2x} + 2$

$du = 2e^{2x}\,dx$

$= \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ln|u| + C$

$= \boxed{\frac{1}{2}\ln|e^{2x}+2| + C}$

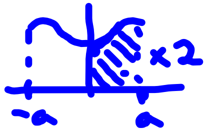
$= \boxed{\ln\sqrt{e^{2x}+2} + C}$

Symmetric Functions:

If f is a continuous, even function on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

symm about y-axis

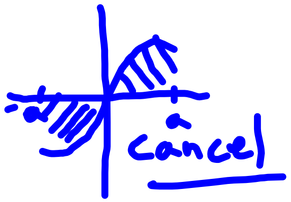
$f(-x) = f(x)$



If f is a continuous, odd function on $[-a, a]$, then $\int_{-a}^{a} f(x)\,dx = 0$

symm about origin

$f(-x) = -f(x)$



Compute $\int x^2\sqrt{x-1}\,dx$

$u = x-1$

$du = 1 \cdot dx$

$x = u+1$

$= \int x^2 \sqrt{u}\, du$

$= \int (u+1)^2 \sqrt{u}\, du$

$= \int (u^2+2u+1)\, u^{1/2}\, du$

$= \int (u^{5/2} + 2u^{3/2} + u^{1/2})\, du$

$= \frac{2}{7} u^{7/2} + 2 \cdot \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C$

$= \boxed{\frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C}$