3.7, 3.9

18. Let C be the graph of

$$x = 4t^3 - 2t^2 + 3t$$
$$y = \frac{1}{4}t^4 - 8t + 7.$$

(i) (4 points) Find a Cartesian equation for the tangent line to C at the point where $t = 1$.

Point: $x = 4(1)^3 - 2(1)^2 + 3(1) = 5$ $\quad (5, -\frac{3}{4})$

$y = \frac{1}{4}(1)^4 - 8(1) + 7 = \frac{-3}{4}$

Slope:
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t^3 - 8}{12t^2 - 4t + 3} \qquad m = \frac{1^3 - 8}{12 - 4 + 3} = \frac{-7}{11}$$

$$\boxed{y + \frac{3}{4} = \frac{-7}{11}(x - 5)}$$

(ii) (2 points) Find a (nonzero) vector tangent to C at the point where $t = 1$

$\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$

$\vec{r}\,'(1) = \boxed{11\vec{i} - 7\vec{j}}$

(iii) (2 points) Find a value of t such that the tangent line to C at that point is horizontal.

$m = 0 \qquad \frac{dy}{dt} = 0$

$t^3 - 8 = 0$

$t = 2$

3.4

44) (ver)

$$\lim_{x\to 0} \frac{(1-\cos x)}{x^2}\frac{(1+\cos x)}{(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} \quad = \sin^2 x$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{1+\cos 0} \quad \frac{1}{2}$$

1

$$= \boxed{\frac{1}{2}}$$

F '06

(3.7) 11. The parametric curve $x = t\left(t^2 - 1\right)$, $y = t^2 - 1$, crosses itself at $(0, 0)$. Find the angle between the two tangent lines at $(0, 0)$.

$t^3 - t$ x   y

(a) 0°

(b) 30°

(c) 45°

(d) 60°

(e) 90°

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ use tangent vectors

Solve for t:

$t(t^2-1)=0$

$t=0$ $t=1,-1$

$t^2-1=0$

$t=1,-1$

$\vec{r}'(t) = (3t^2-1)\vec{i} + (2t)\vec{j}$

$\vec{a} = \vec{r}'(1) = 2\vec{i} + 2\vec{j}$

$\vec{b} = \vec{r}'(-1) = 2\vec{i} - 2\vec{j}$

$\vec{a}\cdot\vec{b} = 0$ so $90°$

F03

(4.2) 10. The function $f(x) = x^5 + x^3 + x$ is one-to-one and so it has an inverse function g. Find $g'(3)$.

a) 6

b) 3

c) 1

d) $\frac{1}{3}$

e) $\frac{1}{9}$

$$g'(3) = \frac{1}{f'(g(3))}$$

$$f'(x) = 5x^4 + 3x^2 + 1$$

$$g'(3) = \frac{1}{f'(1)}$$

$$= \boxed{\frac{1}{9}}$$

$g(3) = y$ means

$f(y) = 3$

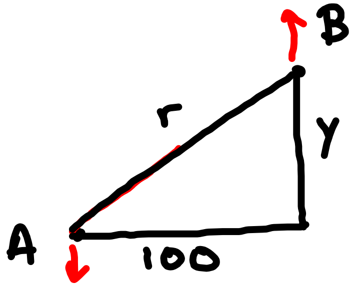
$y^5 + y^3 + y = 3$

$y = 1$ by inspection

$g(3) = 1$

3.10

13)



35+25

$$\frac{dy}{dt} = 60 \ \frac{km}{hr}$$

$$\frac{dr}{dt} = ?$$

$$y = 240 \ km$$

$$r = \sqrt{100^2 + 240^2}$$
$$= 260$$

Pythagoras

$$r^2 = y^2 + 100^2$$

$$2r\frac{dr}{dt} = 2y\frac{dy}{dt}$$

$$2(260)\frac{dr}{dt} = 2(240)(60)$$

$$\frac{dr}{dt} = \boxed{\frac{2(240)(60)}{2(260)} \ \frac{km}{hr}}$$

$$=$$

F07 (4.2)

14. Let $y = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$. Express x in terms of y.

(a) $x = \dfrac{\sqrt{y}-1}{\sqrt{y}+1}$

(b) $x = \dfrac{(1+y)^2}{(1-y)^2}$

(c) $x = \dfrac{1+\sqrt{y}}{1-\sqrt{y}}$

(d) $x = \dfrac{(1-y)^2}{(1+y)^2}$

(e) $x = \dfrac{1+\dfrac{1}{\sqrt{y}}}{1-\dfrac{1}{\sqrt{y}}}$

$$y + y\sqrt{x} = 1 - \sqrt{x}$$

$$y\sqrt{x} + \sqrt{x} = 1 - y$$

$$\sqrt{x}\,(y+1) = 1 - y$$

$$\sqrt{x} = \frac{1-y}{1+y}$$

$$x = \left(\frac{1-y}{1+y}\right)^2$$

F '03

3.4

12. (6 points) Evaluate $\lim_{x\to 0} 12x^2 \cot(2x)\csc x$

$$= \lim_{x\to 0} \frac{12x^2}{x} \cdot \frac{\cos(2x)}{\sin(2x)} \cdot \frac{1}{\frac{\sin x}{x}}$$

$$\frac{12x^2}{x} \cdots \frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{12x}{2x} \cdot \cos 2x \cdot \frac{2x}{\sin 2x} \cdot \frac{x}{\sin x}$$

$$= \lim_{x\to 0} 6 \cdot \cos 2x \cdot \frac{2x}{\sin 2x} \cdot \frac{x}{\sin x}$$

$$= 6 \cdot 1 \cdot 1 \cdot 1$$

Key Limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{x}{\sin x} = 1$$

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

F'02 (3.11) $f(a)+f'(a)(x-a)$

17. Suppose F and G are differentiable functions. The line $y = 1 + 2x$ is the tangent-line approximation to F at $x = 2$, whereas the line $y = 2 - 3x$ is the tangent-line approximation to G at $x = 2$.

(i) (4 points) Find $F(2)$, $F'(2)$, $G(2)$, and $G'(2)$.

F: $F(2) + F'(2)(x-2) = 1+2x$

$F(2) + 2(x-2) = 1+2x$

$(F(2)-4) + 2x = 1+2x$

$\boxed{F'(2) = 2 \quad F(2) = 5}$

G: $\boxed{G'(2) = -3}$

$G(2) + -3(x-2) = 2-3x$

$G(2) - 3x + 6 = 2 - 3x$

$\boxed{G(2) = -4}$

(ii) (6 points) Find the tangent-line approximation to $\frac{F}{G} = H$ at $x = 2$.

$H(2) + H'(2)(x-2)$

$H(2) = \frac{F(2)}{G(2)} = \frac{5}{-4} = \frac{-5}{4}$

$H'(2) = \frac{G(2)F'(2) - F(2)G'(2)}{(G(2))^2} =$

F '07

3.11 dC C

18. [8 points] The circumference of a sphere (the length of its "equatorial circle") was measured to be 20 cm with a possible error of 1 cm. Estimate the maximum error in the calculated volume of the sphere using differentials. Recall that the volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the circumference of a circle is $C = 2\pi r$. Here r is the radius.

eliminate r:

$$r = \frac{C}{2\pi}$$

$$V = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3$$

$$= \frac{\cancel{4}^{1}}{3\cancel{\pi}}\frac{C^3}{\cancel{8}_{2}\pi^{\cancel{3}\,2}}$$

$$V = \frac{1}{6\pi^2}C^3$$

$$dV = \frac{1}{6\pi^2}(3C^2)\,dC$$

$$dV = \frac{1}{2\pi^2}(20)^2(1) = \boxed{\frac{200}{\pi^2}\text{ cm}^3}$$

F'06 (4.2)

7. Given that $f(x) = \sqrt{1-x}$, find the range of f^{-1}, the inverse of f.

(a) $(-\infty, \infty)$

(b) $[1, \infty)$

(c) $(-\infty, 1]$

(d) $[-1, 1]$

(e) $[0, \infty)$

= domain of f

all x such that $1 - x \geq 0$

$1 \geq x$

$x \leq 1$ $(-\infty, 1]$

(Extra: Find inverse and domain)

$x = \sqrt{1-y}$

$x^2 = 1 - y$

$y = 1 - x^2$; $x \geq 0$ (range of f)