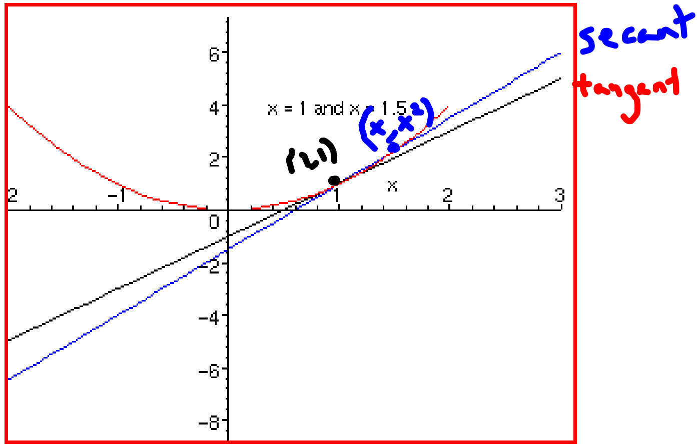
2.7: Tangents, Velocities, and Rates of Change

We are now ready to find a formal way of computing the slope of a line tangent to a curve at a point. Re-view the animation from 2.1 posted on my webpage. What happens as the second x coordinate moves closer to the given tangent-line point?

The slopes of secant lines get closer to the slope of the tangent line.



Each secant line above passes through the point (1, 1). If x is the x-coordinate of the second point, write an expression for the slope of the line between the two points.

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{x^2 - 1}{x - 1}$$

Write and solve a limit problem which allows us to find the slope of the tangent line at $x = 1$.

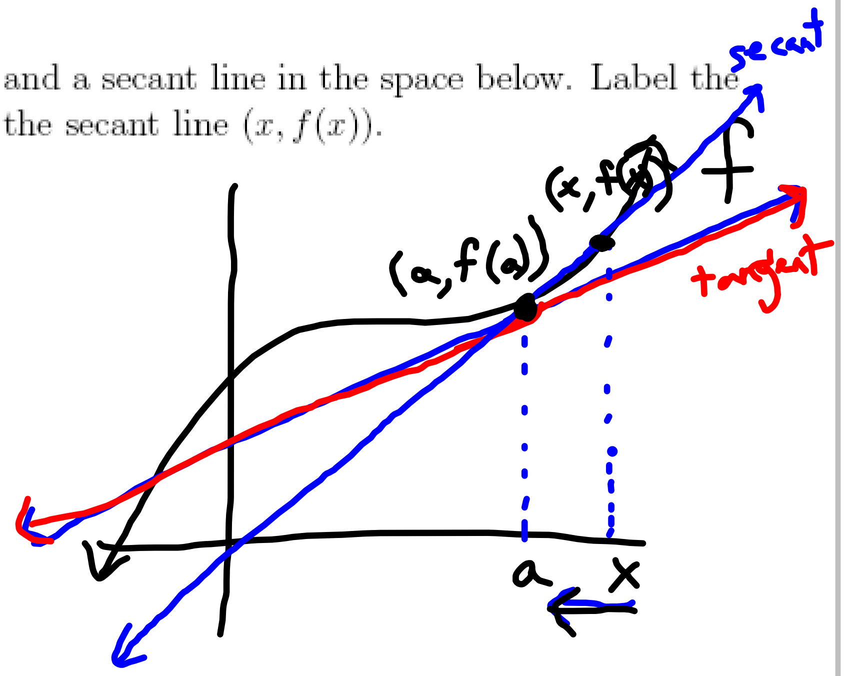
of $f(x) = x^2$

$$m_{tan} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = 1 + 1 = \boxed{2}$$

More General: Draw any function, a tangent line, and a secant line in the space below. Label the tangent-line point $(a, f(a))$ and the second point on the secant line $(x, f(x))$.

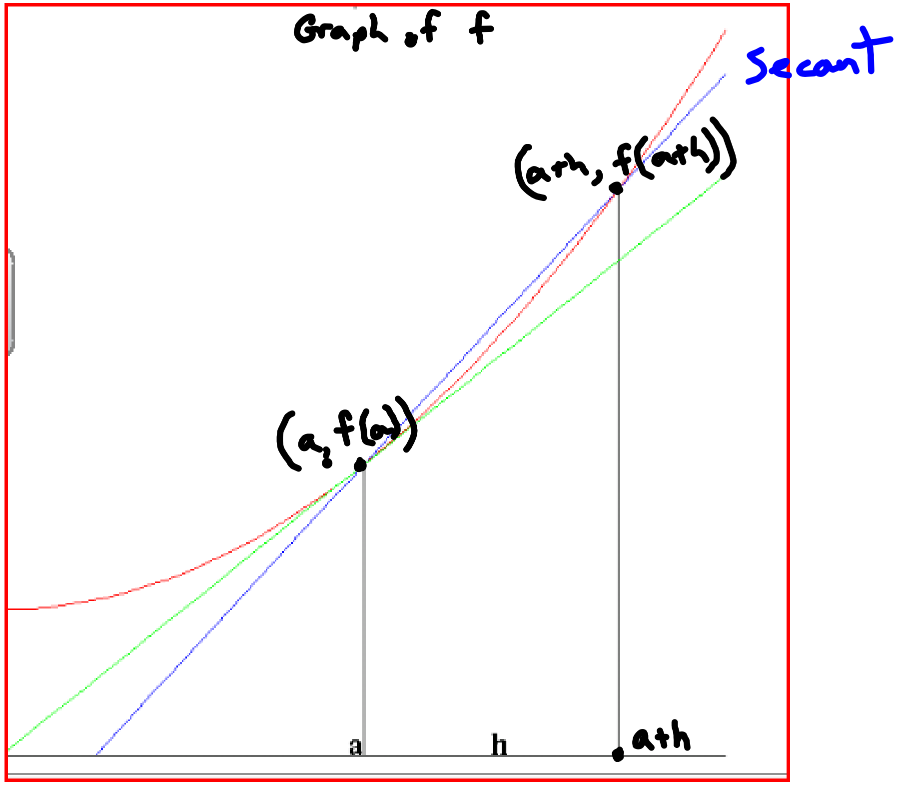
$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$



What should happen as the point $(x, f(x))$ moves closer to the tangent-line point $(a, f(a))$? Write a limit which explains this mathematically:

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Most General: Draw any function, a tangent line, and a secant line again in the space below. Label the tangent-line point $(a, f(a))$ and let h be the distance bewteen the x values on the secant line. View the new animation posted under today's notes for a visual understanding of this).



$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{(a+h)-a}$$

$$= \frac{f(a+h)-f(a)}{h}$$

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

"f prime of a"

The *derivative* of a function at $x = a$ is given by $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

OR $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

Examples: Use a limit definition to find the equation of the line tangent to the curve $f(x) = x^2 - 4x + 4$ at the point where $x = 3$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$$

1 *

$$= \lim_{x \to 3} \frac{(x^2 - 4x + 4) - (3^2 - 4 \cdot 3 + 4)}{x - 3}$$

$$= \lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{x-3}} = 3 - 1 = 2$$

slope $= 2$, point $(3, 1^{*})$

Eqn: Point-Slope $y - y_1 = m(x - x_1)$

$$\boxed{y - 1 = 2(x - 3)}$$

OR

$y = mx + b$

$1 = 2 \cdot 3 + b$

$b = -5$

$$\boxed{y = 2x - 5}$$

Use a limit definition to find the derivative of the function $f(x) = \frac{1}{x-2}$. ~~Compute the slopes of the lines tangent to this graph~~ at $x = 0$, $x = 1$, and $x = 3$.

Use $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ at $x = \underline{a}$

$$= \lim_{h\to 0} \frac{\frac{1}{a+h-2} - \frac{1}{a-2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(a+h-2)}\frac{a-2}{(a-2)} - \frac{1}{(a-2)}\frac{(a+h-2)}{(a+h-2)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{(a-2) + -(a \overset{-}{+} h + 2)}{(a+h-2)(a-2)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{\cancel{a} \cancel{-2} \cancel{-a} - h \cancel{+2}}{(a+h-2)(a-2)}\right)$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}} \cdot \frac{-\overset{1}{\cancel{h}}}{(a+\underset{0}{\cancel{h}}-2)(a-2)} = \frac{-1}{(a-2)^2}$$

at $x=0$: $f'(0) = \frac{-1}{(0-2)^2} = \boxed{-\frac{1}{4}}$

$x=1$: $f'(1) = \frac{-1}{(1-2)^2} = \boxed{-1}$

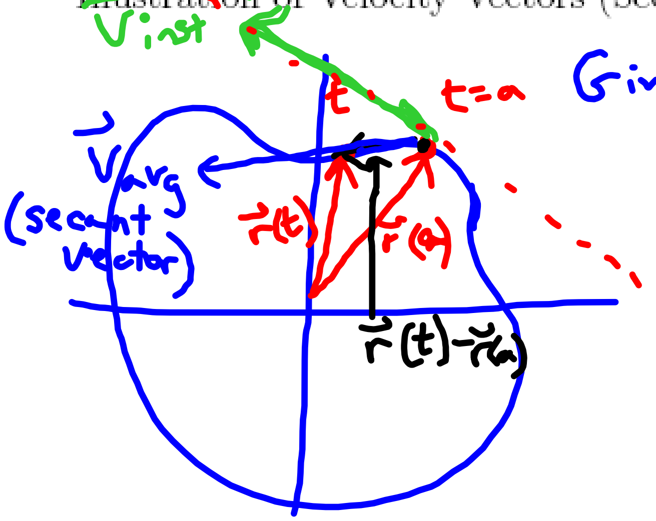
$x=3$: $f'(3) = \frac{-1}{(3-2)^2} = \boxed{-1}$

tangent line ($\vec r'(a)=\vec v$ in $\vec r_0+t\vec v$)

Secant and Tangent Vectors-an Introduction

(tangent vector)

Illustration of Velocity Vectors (Secant and Tangent):



Given $\vec r(t)$ is position

$$\vec V_{avg} = \frac{\Delta \vec r}{\Delta t} = \frac{\vec r(t)-\vec r(a)}{t-a}$$

As $t\to a$, $t-a<1$

$$\vec V_{inst} = \lim_{t\to a} \frac{\vec r(t)-\vec r(a)}{t-a}$$

OR $$\vec r\,'(a) = \lim_{h\to 0} \frac{\vec r(a+h)-\vec r(a)}{h}$$

derivative

Example: Find a vector tangent to the curve $\mathbf{r}(t) = \underline{\overset{\times}{(3t^2)}\mathbf{i} + \sqrt{t}\,\mathbf{j}}$ at the point $\underline{(3,1)}$. Then find parametric equations of the line tangent to the curve at this point.

$$\vec{r}\,'(a) = \lim_{t\to a} \frac{\vec{r}(t) - \vec{r}(a)}{t-a}$$

Problem: What is a?

$3t^2 = 3$ and $\sqrt{t} = 1$

$t = \cancel{\pm} 1$ $\qquad t = 1$

Must satisfy BOTH equations

$a = 1$

$$\vec{r}\,'(1) = \lim_{t\to 1} \frac{\vec{r}(t) - \vec{r}(1)}{t-1}$$

$$= \lim_{t\to 1} \frac{(3t^2\vec{i} + \sqrt{t}\,\vec{j}) - (3\vec{i} + \vec{j})}{t-1}$$

$$= \lim_{t\to 1} \frac{(3t^2-3)\vec{i} + (\sqrt{t}-1)\vec{j}}{t-1}$$

$$= \lim_{t\to 1} \left(\frac{3t^2-3}{t-1}\right)\vec{i} + \left(\frac{\sqrt{t}-1}{t-1}\right)\vec{j}$$

$$= \left(\lim_{t\to 1} \frac{3t^2-3}{t-1}\right)\vec{i} + \left(\lim_{t\to 1} \frac{(\sqrt{t}-1)(\sqrt{t}+1)}{(t-1)(\sqrt{t}+1)}\right)\vec{j}$$

$$= \left(\lim_{t\to 1} \frac{3\cancel{(t-1)}(t+1)}{\cancel{t-1}}\right)\vec{i} + \left(\lim_{t\to 1} \frac{\cancel{t-1}}{\cancel{(t-1)}(\sqrt{t}+1)}\right)\vec{j}$$

$$\vec{r}\,'(1) = \boxed{6\vec{i} + \tfrac{1}{2}\vec{j}}$$

Parametric: $\vec{r}_0 + t\vec{v}$ $\qquad \vec{r}_0 = 3\vec{i} + \vec{j}$ (the point)

$\vec{v} = 6\vec{i} + \frac{1}{2}\vec{j}$ (the velocity/derivative)

$$\vec{r}(t) = (3\vec{i} + \vec{j}) + t(6\vec{i} + \tfrac{1}{2}\vec{j})$$

$$= (3\vec{i} + \vec{j}) + (6t\,\vec{i} + \tfrac{1}{2}t\,\vec{j})$$

$$= \boxed{(3+6t)\vec{i} + (1+\tfrac{1}{2}t)\vec{j}}$$

OR

$x = 3 + 6t$

$y = 1 + \frac{1}{2}t$