3.1: The Derivative

Now that we can find the slope of the line tangent to a curve at any point (provided the limit of the slopes exists), we can talk about a new function based on this calculation.

Definition: The *derivative function* of a function f (or the *derivative of f*) is a function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>Old function:</u>

Input x-value $\longrightarrow \boxed{f} \longrightarrow$ output $y = f(x)$ (y-value)

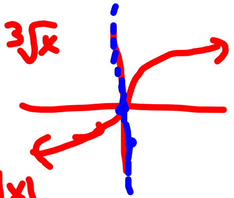
<u>New function:</u>

Input x-value $\longrightarrow \boxed{f'} \longrightarrow f'(x)$ slope of tangent line at $(x, f(x))$

When is f not <u>*differentiable*</u>? (i.e., when does $f'(x)$ not exist or when is x not in the domain of f'?

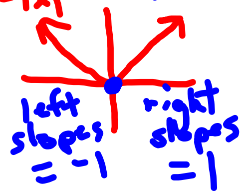
1) infinite limit $\left(\frac{\#}{0}\right)$ vertical tangent line

$f(x) = \sqrt[3]{x}$



2) left limit $\neq$ right limit sharp corner

$f(x) = |x|$



3) f not cts

Examples:

Let $f(x) = \dfrac{1}{1+2x}$. Find $f'(x)$ using the limit definition and use it to determine the slope of the line tangent to f at $x = 0$.

NOT $\cancel{\dfrac{1}{1+2x} + h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{1+2(x+h)} - \dfrac{1}{1+2x}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{1}{(1+2x+2h)}\frac{(1+2x)}{(1+2x)} - \frac{1}{(1+2x)}\frac{(1+2x+2h)}{(1+2x+2h)} \right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{(1+2x) - (1+2x+2h)}{(1+2x+2h)(1+2x)} \right)$$

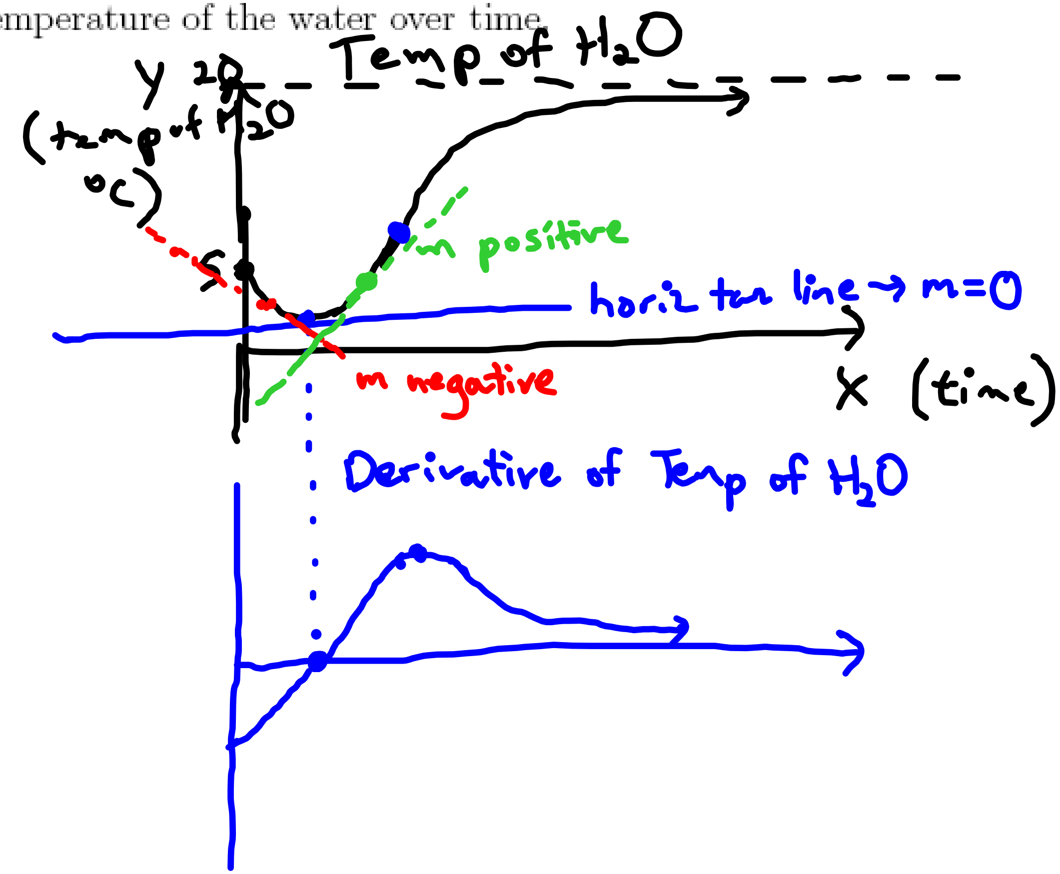
$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\frac{\cancel{1}+\cancel{2x}-\cancel{1}-\cancel{2x}-2\cancel{h}}{(1+2x+2h)(1+2x)} \right)$$

$$= \lim_{h\to 0} \frac{-2}{(1+2x+\underset{0}{\cancel{2h}})(1+2x)} = \boxed{\frac{-2}{(1+2x)^2} = f'(x)}$$

at $x=0$: $\quad m = f'(0) = \dfrac{-2}{(1+2\cdot 0)^2} = \boxed{-2}$

Ice cubes are placed in a glass, then at time $t = 0$ the glass is filled with water from a $5°C$ refrigerator. If the glass sits on a table in a $20°C$ room, sketch a rough graph of the temperature of the water, in degrees Celsius, over time. Then sketch the graph of the rate of change of the temperature of the water over time.

derivative



Determine whether the function below is differentiable at $x = 1$:

$$f(x) = \begin{cases} 6x & \text{if } x \geq 1 \\ 3x^2 + 2 & \text{if } x < 1 \end{cases}$$

(differentiable everywhere else since polynomials diff everywhere)

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

(since only care about one point)

left

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{(3x^2 + 2) - 6}{x - 1}$$

$$\frac{-1 \, (-)}{0 \, (-)}$$

$$= +\infty$$

right

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{6x - 6}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{6\cancel{(x-1)}}{\cancel{x-1}} = 6$$

$\therefore$ f is not differentiable at $x=1$

OR $f'(1)$ DNE