3.4: Limits and Derivatives of Trig Functions

Key Limit: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

"Proof": (zoom in on graph of $y = \sin x$ at $x = 0$)

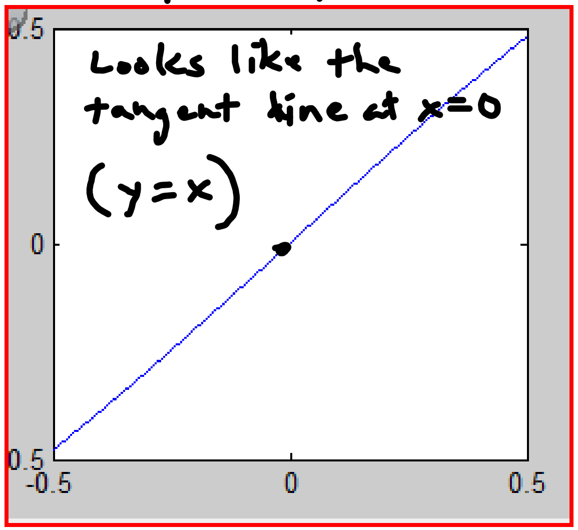
Implication: $f'(0) = 1$

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 1$$

$$\lim_{x\to 0} \frac{\sin x - 0}{x - 0} = 1$$

$$* \quad \lim_{x\to 0} \frac{\sin x}{x} = 1 \quad *$$

$y = \sin x$



Key Limit: $\lim_{x\to 0} \frac{(\cos x - 1)}{x} \cdot \frac{(\cos x + 1)}{(\cos x + 1)} =$

$\sin^2 x + \cos^2 x = 1$

$\cos x = \sqrt{1 - \sin^2 x}$

Proof:

$$\lim_{x\to 0} \frac{\overbrace{\cos^2 x - 1}^{= -\sin^2 x}}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{-\sin x \cdot \sin x}{x(\cos x + 1)} = \frac{-\sin 0}{\cos 0 + 1} = \boxed{0}$$

$$\therefore \quad * \quad \boxed{\lim_{x\to 0} \frac{\cos x - 1}{x} = 0} \quad *$$

We can use these limits to find the derivative of $f(x) = \sin x$ using the definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right)$$

$$= \lim_{h\to 0} \sin x \left(\underbrace{\frac{\cos h - 1}{h}}_{0} \right) + \cos x \underbrace{\frac{\sin h}{h}}_{1}$$

$$f'(x) = \cos x$$

$$* \boxed{\frac{d}{dx}(\sin x) = \cos x} *$$

Similarly, we can show that $\frac{d}{dx}(\cos x) = -\sin x$

Once we know these, we can find the derivatives of all the other trig functions using quotient rules:

Example: $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$ HI LO

$$= \frac{\cos x \cdot \cos x + \sin x(+\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$* \boxed{\frac{d}{dx}(\tan x) = \sec^2 x} *$$

Other derivatives:

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Examples:

$$\tan 5x = \frac{\sin 5x}{\cos 5x}$$

Compute $\lim_{x \to 0} \frac{2x}{\tan 5x}$

$$= \lim_{x \to 0} \frac{2\cancel{x}}{5\cancel{x}} \cdot \cos 5x \cdot \frac{1}{\frac{\sin 5x}{5x}} \to 1$$

$$= \frac{2}{5} \cdot 1 \cdot 1$$

$$= \boxed{\frac{2}{5}}$$

Notes:

$$\lim_{x \to 0} \frac{x}{\sin x} = 1$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

Differentiate $f(x) = \dfrac{1 - \cos x^{\text{HI}}}{\sin x \text{ LO}}$ and $g(x) = \dfrac{\sin x^{\text{HI}}}{1 + \cos x_{\text{LO}}}$ and simplify

$$f'(x) = \frac{\sin x(\sin x) - (1-\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{1 - \cos x}{\sin^2 x\,(=1-\cos^2 x)}$$

$$= \frac{\cancel{1 - \cos x}}{(\cancel{1-\cos x})(1+\cos x)}$$

$$= \frac{1}{1+\cos x}$$

$$g'(x) = \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{1 + \cos x}{(1+\cos x)^2}$$

$$= \frac{1}{1+\cos x}$$

NOTE:

$$\frac{1-\cos x}{\sin x} \ne \frac{\sin x}{1+\cos x}$$

$$1 - \cos^2 x = \sin^2 x$$

Determine when the graph of $y = \cancel{\sec x} \tan^2 x$, $0 \leq x \leq 2\pi$ has a horizontal tangent line

$$y = \tan x \cdot \tan x$$ Product Rule

$$y' = \tan x \cdot \sec^2 x + \tan x \cdot \sec^2 x = 0$$

$$2 \tan x \sec^2 x = 0$$

$$\tan x = 0$$

$$\boxed{x = 0, \pi, 2\pi}$$

$$\sec^2 x = 0$$

$$\frac{1}{\cos^2 x} = 0$$

$$1 = 0$$