3.5: The Chain Rule

We know $\frac{d}{dx}(\sin x) = \cos x$. Does $\frac{d}{dx}(\sin(2x)) = \cos(2x)$?
(HINT: Rewrite the function as an identity, then differentiate).

$$\frac{d}{dx}(\sin 2x) = \frac{d}{dx}(2\sin x \cos x)$$ Product Rule

$$= 2\sin x \cdot (-\sin x) + \cos x(2\cos x)$$

$$= 2(\cos^2 x - \sin^2 x)$$

$$= 2\cos(2x)$$

Recall: The *composition* of 2 functions f and g is defined by $(f \circ g)(x) = f(\underline{g(x)})$

inside parentheses

Define f and g for the above function.

$\sin(2x)$ $\quad$ $f(x) = \sin x$ (outside)

$g(x) = 2x$ (inside)

The Chain Rule: If f and g are differentiable functions, $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

An alternate version of the Chain Rule states that $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Examples:

Find the derivatives of the following:

(Multiple Chain Rule)
Outside → In

$f(x) = \underbrace{(x^3 - 4)}_{\text{stuff in parentheses}}^{10}$

$f'(x) = 10(x^3 - 4)^9 \cdot \underbrace{3x^2}_{\text{deriv of stuff}}$

$y = \cos^3(2x)$

$y = (\underbrace{\cos(2x)}_{\text{stuff}})^3$

$y' = 3(\cos(2x))^2 \cdot \frac{d}{dx}(\cos(\underbrace{2x}_{\text{More stuff}}))$

$y' = 3\cos^2(2x)(-\sin(2x)) \cdot \underbrace{2}_{\text{deriv of more stuff}}$

$= \boxed{-6\cos^2(2x)\sin(2x)}$

Differentiate $y = \dfrac{(3x-2)^4}{(2x+5)^3}$ two different ways.

HI

LO

① Quotient Rule

$$y' = \frac{(2x+5)^3(4)(3x-2)^3(3) - (3x-2)^4(3)(2x+5)^2(2)}{(2x+5)^6}$$

Factor out smallest exponents

$$= \frac{(3)(2)\cancel{(2x+5)^2}(3x-2)^3\left[\overset{4x+10}{(2x+5)(2)} - \overset{-3x+2}{(3x-2)}\right]}{(2x+5)^{\cancel{6}\,4}}$$

$$= \boxed{\frac{6(3x-2)^3(x+12)}{(2x+5)^4}}$$

② Product Rule

$$y = (3x-2)^4(2x+5)^{-3}$$

$$y' = (3x-2)^4(-3)(2x+5)^{-4}(2) + (2x+5)^{-3}(4)(3x-2)^3(3)$$

$$= 3(2)(3x-2)^3(2x+5)^{-4}\left[(3x-2)(-1) + (2x+5)^1(2)\right]$$

$$= \boxed{\frac{6(3x-2)^3(x+12)}{(2x+5)^4}}$$

Given functions f and g such that $f(4) = 2$, $f'(4) = 2$, $g(1) = 4$, $g'(1) = 2$, find $h'(1)$ if $h(x) = f(g(x))$

$h(x) = f(g(x))$ $\qquad$ $h'(1)$?

$h'(x) = f'(g(x)) \cdot g'(x)$

plug in $x=1$

$h'(1) = f'(g(1))\, g'(1)$

$= f'(4)\,(2)$

$= 2 \cdot 2$

$\boxed{h'(1) = 4}$