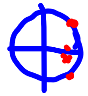
3.6-Implicit Differentiation

$x^2 + y^2 - 1 = 0$



The equation $F(x, y) = 0$ *implicitly* defines a relation (not necessarily a function) between y and x. The *graph* of $F(x, y) = 0$ is the set of all points (x, y) such that the equation holds ($\{(x, y)|F(x, y) = 0\}$). Given a graph of an implicitly-defined relation, we can still talk about the slope of the line tangent to the curve at a given point.

Method for Implicit Differentiation:

1. Done when y is not explicitly defined as a function of x.
2. Differentiate both sides of the equation, remembering that y depends on x (can call it $y(x)$)
3. Solve for $y'(x)$ or $\frac{dy}{dx}$

Examples:

Find $\frac{dy}{dx}$ implicitly if $x^2y + 2x = 9y + 4$. Then solve for y and show you get the same answer.

$$x^2 y(x) + 2x = 9y(x) + 4$$

Product Rule

$$x^2 \cdot y'(x) + y(x) \cdot 2x + 2 = 9y'(x)$$

$$x^2y' + 2xy + 2 = 9y'$$

$$x^2y' - 9y' = -2xy - 2$$

$$y'(x^2-9) = -2xy - 2$$

$$\boxed{y' = \frac{-2xy-2}{x^2-9}}$$

$$y' = \frac{\left[-2x\left(\frac{-2x+4}{x^2-9}\right) - 2\right] x^2-9}{(x^2-9)\;(x^2-9)}$$

$$= \frac{-2x(-2x+4) - 2(x^2-9)}{(x^2-9)^2}$$

$$x^2y - 9y = -2x + 4$$

$$y(x^2-9) = -2x + 4$$

$$y = \frac{-2x+4}{x^2-9} \quad \begin{matrix}\text{HI}\\ \text{LO}\end{matrix}$$

$$\frac{dy}{dx} = \frac{(x^2-9)(-2) - (-2x+4)(2x)}{(x^2-9)^2}$$

$$= \frac{-2x^2 + 18 + 4x^2 - 8x}{(x^2-9)^2}$$

$$= \boxed{\frac{2x^2 - 8x + 18}{(x^2-9)^2}}$$

Find $\frac{dy}{dx}$ if $x^2y^2 = 2(x^2 + y^2)$

$$x^2(y(x))^2 = 2x^2 + 2(y(x))^2$$

$$x^2 \cdot 2\, y(x)\, y'(x) + (y(x))^2 \cdot 2x = 4x + 4\, y(x)\, y'(x)$$

$$2x^2yy' + 2xy^2 = 4x + 4yy'$$

$$2x^2yy' - 4yy' = 4x - 2xy^2$$

$$y'(2x^2y - 4y) = 4x - 2xy^2$$

$$\boxed{y' = \frac{4x - 2xy^2}{2x^2y - 4y}}$$

Find the slope of the line tangent to $\sec(x+y) - \tan(x-y) = 1$ at the point (π, π)

$$\sec(x+y(x)) - \tan(x-y(x)) = 1$$

$$\sec(x+y(x))\tan(x+y(x))\cdot(1+y'(x)) - \sec^2(x-y(x))(1-y'(x)) = 0$$

$$\underbrace{\sec(x+y)\tan(x+y)}_{\text{distribute}}(1+y') - \underbrace{\sec^2(x-y)}_{\text{distribute}}(1-y') = 0 \qquad \text{Multiply out}$$

$$\sec(x+y)\tan(x+y)(1) + \sec(x+y)\tan(x+y)\,y' - \sec^2(x-y)(1) + \sec^2(x-y)\,y' = 0$$

$$\sec(x+y)\tan(x+y)\,y' + \sec^2(x-y)\,y' = -\sec(x+y)\tan(x+y) + \sec^2(x-y)$$

$$y' = \frac{-\sec(x+y)\tan(x+y) + \sec^2(x-y)}{\sec(x+y)\tan(x+y) + \sec^2(x-y)}$$

@ (π, π) $\quad y' = \dfrac{-\sec(\pi+\pi)\tan(\pi+\pi) + \sec^2(\pi-\pi)}{\sec(\pi+\pi)\tan(\pi+\pi) + \sec^2(\pi-\pi)} = \boxed{1}$

Alternative: Substitute point, THEN SOLVE:

$\vdots$

$$\sec(x+y)\tan(x+y)(1+y') - \sec^2(x-y)(1-y') = 0 \qquad \overset{x}{(\pi},\overset{y}{\pi)}$$

$$\cancel{\sec 2\pi \tan 2\pi}(1+y') - \sec^2 0\,(1-y') = 0$$

$$-1 + y' = 0$$

$$y' = \boxed{1}$$

Show that the curves $x^2 + y^2 = 9$ and $y = \sqrt{2}x$ are orthogonal. $\perp$

(show slopes of tangent lines are negative reciprocals)

$$2x + 2y\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \sqrt{2}$$

$$\cancel{2}y\frac{dy}{dx} = -\cancel{2}x$$

$$\frac{dy}{dx} = \frac{-x}{y}.$$

? neg recip?

Method I: Find intersection point(s) and substitute into derivative.

Method II: intersect (x,y) solves both equations:

$$\frac{dy}{dx} = \frac{-x}{y} \qquad y = \sqrt{2}\,x$$

$$= \frac{-x}{\sqrt{2}\,x} = \frac{-1}{\sqrt{2}} \qquad \sqrt{2}$$

negative reciprocals, so curves are $\perp$.