## 3.7-Derivatives of Vector Functions

Recall definition:

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \qquad \text{if } \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

$$= \lim_{h\to 0} \frac{\left[x(t+h)\vec{i} + y(t+h)\vec{j}\right] - \left[x(t)\vec{i} + y(t)\vec{j}\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[x(t+h) - x(t)\right]\vec{i} + \left[y(t+h) - y(t)\right]\vec{j}}{h}$$

$$= \lim_{h\to 0} \left( \frac{x(t+h) - x(t)}{h}\vec{i} + \frac{y(t+h) - y(t)}{h}\vec{j} \right)$$

$$= \left( \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} \right)\vec{i} + \left( \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \right)\vec{j}$$

$$* \boxed{\vec{r}\,'(t) = x'(t)\,\vec{i} + y'(t)\,\vec{j}} *$$

What the derivative of a vector function tells us:

1) a vector tangent to curve
2) a direction vector for tangent line $\qquad \vec{r}_0 + t\boxed{\vec{v}}$
3) if $\vec{r}$ = position, $\vec{r}\,'$ = velocity

*Examples:*

Given the position function of a particle is $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$, find the position and velocity at $t = 0$. What does this tell you about the motion of the particle? Find the speed of the particle at the point where $t = \frac{\pi}{4}$.

Ellipse

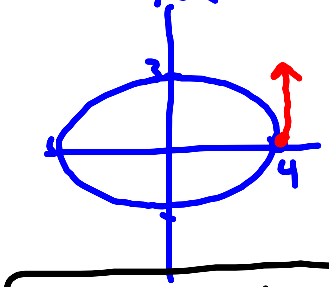
position: $\vec{r}(0) = (4\overset{1}{\cos 0})\vec{i} + (3\overset{0}{\sin 0})\vec{j}$

$= 4\vec{i} + 0\vec{j} \qquad (4,0)$

velocity: $\vec{r}\,'(t) = (-4\sin t)\vec{i} + (3\cos t)\vec{j}$

$\vec{r}\,'(0) = (-4\overset{0}{\sin 0})\vec{i} + (3\overset{1}{\cos 0})\vec{j}$

$= \boxed{0\vec{i} + 3\vec{j}}$



Counter clockwise

speed $= |\vec{r}\,'(\frac{\pi}{4})|$

$\vec{r}\,'(\frac{\pi}{4}) = (-4\overset{\frac{\sqrt{2}}{2}}{\sin\frac{\pi}{4}})\vec{i} + (3\overset{\frac{\sqrt{2}}{2}}{\cos\frac{\pi}{4}})\vec{j}$

$= \underline{-2\sqrt{2}\vec{i} + \frac{3\sqrt{2}}{2}\vec{j}}$

Speed $= |\vec{r}\,'(\frac{\pi}{4})| = \boxed{\sqrt{(-2\sqrt{2})^2 + \left(\frac{3\sqrt{2}}{2}\right)^2}} = \sqrt{\frac{8\cdot 4}{4} + \frac{18}{4}} = \sqrt{\frac{50}{4}} = \boxed{\frac{5\sqrt{2}}{2}}$

$(\tan t)^2$

Find a unit tangent vector for the curve $\mathbf{r}(t) = (\sec t)\mathbf{i} + (\tan^2 t)\mathbf{j}$ at the point where $t = \frac{\pi}{3}$.

deriv

$$\vec{r}\,'(t) = (\sec t \tan t)\vec{i} + (2\tan t \sec^2 t)\vec{j}$$

$$\vec{r}\,'\left(\tfrac{\pi}{3}\right) = \left(\sec\tfrac{\pi}{3}\tan\tfrac{\pi}{3}\right)\vec{i} + \left(2\tan\tfrac{\pi}{3}\sec^2\tfrac{\pi}{3}\right)\vec{j}$$

$$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{1/2} = 2 \qquad \tan\frac{\pi}{3} = \frac{\sin\pi/3}{\cos\pi/3} = \frac{\sqrt{3}/\cancel{2}}{1/\cancel{2}} = \sqrt{3}$$

$$= 2\sqrt{3}\,\vec{i} + 2\sqrt{3}\cdot 4\,\vec{j}$$

$$= 2\sqrt{3}\,\vec{i} + 8\sqrt{3}\,\vec{j}$$

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{1}{\sqrt{(2\sqrt{3})^2 + (8\sqrt{3})^2}}\left(2\sqrt{3}\,\vec{i} + 8\sqrt{3}\,\vec{j}\right)$$

unit tangent vector

$$= \frac{1}{\sqrt{12+192}}\left(2\sqrt{3}\,\vec{i} + 8\sqrt{3}\,\vec{j}\right)$$

$$= \frac{1}{\sqrt{204}}\left(2\sqrt{3}\,\vec{i} + 8\sqrt{3}\,\vec{j}\right)$$

$$= \boxed{\frac{2\sqrt{3}}{\sqrt{204}}\vec{i} + \frac{8\sqrt{3}}{\sqrt{204}}\vec{j}}$$

The graphs of $\underset{1}{\mathbf{r}}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ and $\underset{2}{\mathbf{r}}(t) =< \sqrt{2}\cos t, \sqrt{2}\sin t >$ intersect at the point $(\overset{x}{1},\overset{y}{1})$. Find the angle of intersection to the nearest degree.

Find the angle between the tangent vectors, using

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\vec{r}_1{}'(t) = 2t\vec{i} + 3t^2\vec{j}$$

$$\vec{r}_2{}'(t) = (-\sqrt{2}\sin t)\vec{i} + (\sqrt{2}\cos t)\vec{j}$$

Find t given x and y (set orig eqns = coordinates)

x: $t^2 = 1$ y: $t^3 = 1$

$t = \cancel{\pm}1$ $t = 1$

$$\vec{r}_1{}'(1) = 2\vec{i} + 3\vec{j} = \vec{a}$$

x: $\sqrt{2}\cos t = 1$ y: $\sqrt{2}\sin t = 1$

$\cos t = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\sin t = \frac{1}{\sqrt{2}}$

$t = \frac{\pi}{4}, \cancel{-\frac{\pi}{4}}$ $t = \frac{\pi}{4}, \cancel{\frac{3\pi}{4}}$

$$\vec{r}_2{}'\left(\frac{\pi}{4}\right) = \left(-\sqrt{2}\sin\frac{\pi}{4}\right)\vec{i} + \left(\sqrt{2}\cos\frac{\pi}{4}\right)\vec{j}$$

$$= -1\vec{i} + 1\vec{j} = \vec{b}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(2)(-1) + 3(1)}{\sqrt{4+9}\cdot\sqrt{1+1}} \approx \frac{1}{\sqrt{26}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{1}{\sqrt{26}}\right) \approx 79^\circ}$$