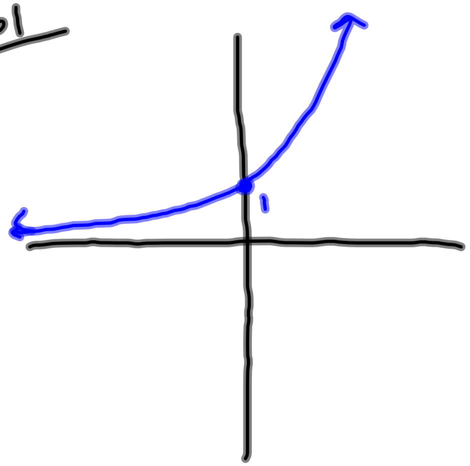
4.1: Exponential Functions

Variable in Exponent

Definition: An *exponential function* is a function of the form $f(x) = a^{x}$, $a > 0$.

Graph and Graphical Properties of $f(x) = a^x$:

$a > 1$



Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Intercept: $(0,1)$

Asymptote: $y=0$

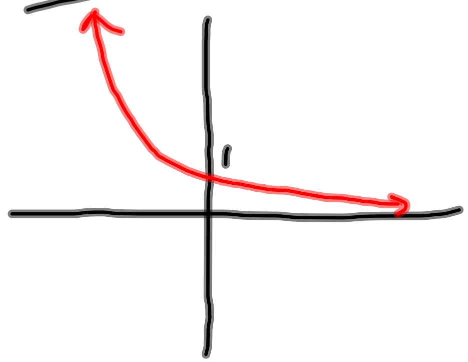
$\lim_{x \to -\infty} a^x = 0$

$\lim_{x \to \infty} a^x = \infty$

Cts

Diff

$a < 1$



Properties of exponential functions:

$$a^x \cdot a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = a^{xy}$$

$$a^{-x} = \frac{1}{a^x}$$

Note: suppose $b<1$

Let $a = \frac{1}{b}$, $a > 1$

$$a^x = \left(\frac{1}{b}\right)^x = \frac{1^x}{b^x} = \frac{1}{b^x} = b^{-x}$$

reflect about y-axis

Using the definition of the derivative, we see that if $f(x) = a^x$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h} \\
&= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \\
&= a^x \left(\lim_{h \to 0} \frac{a^{0+h} - a^0}{h} \right) = a^x \cdot \underline{f'(0)}
\end{aligned}
$$

would be nice if this $= 1$

Definition: e is the number such that

$$f'(0) = 1$$

$e \approx 2.8182818726\ldots$

$$\frac{d}{dx}\left(e^x\right) = e^x \cdot 1 = e^x$$

Examples:

Compute the following limits:

$$\lim_{x \to -4^-} e^{1/(x+4)}$$

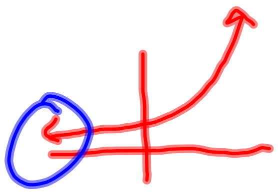
Key: exponential is cts, so

$$\lim_{x \to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$$

$$= e^{\lim_{x\to -4^-} \frac{1}{x+4}} \qquad \frac{1}{0} \quad \pm\infty \text{ which?}$$

$$= e^{\lim_{x\to -4^-} \frac{1}{x+4}} \qquad \frac{+}{-}$$

$$= \text{"} e^{-\infty} \text{"}$$



$$= \boxed{0}$$

$$\frac{4^x}{4^{-x}} = 4^{x-(-x)}$$

$$\lim_{x \to -\infty} \frac{4^x + 4^{-x}}{4^x - 4^{-x}}$$

Factor out 4^{-x} (dominating term)

$$= \lim_{x\to -\infty} \frac{\cancel{4^{-x}}(4^{2x} + 1)}{\cancel{4^{-x}}(4^{2x} - 1)}$$

$$= \lim_{x\to -\infty} \frac{\cancel{4^{2x}}^{\,0} + 1}{\cancel{4^{2x}}_{\,0} - 1} = \boxed{-1}$$

Mar 9-11:13 AM

Find the first and second derivatives of $f(x) = e^{\sqrt{x^2+1}}$ $(x^2+1)^{1/2}$

$$f'(x) = e^{\sqrt{x^2+1}} \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x) = \boxed{\frac{x}{\sqrt{x^2+1}}\, e^{\sqrt{x^2+1}}}$$

Product Rule

$x(x^2+1)^{-1/2}$

$$f''(x) = \frac{x}{\sqrt{x^2+1}} \cdot e^{\sqrt{x^2+1}} \cdot \frac{x}{\sqrt{x^2+1}} + \Big(\underbrace{x\left(-\tfrac{1}{2}\right)(x^2+1)^{-3/2}(2x) + 1(x^2+1)^{-1/2}}\Big) e^{\sqrt{x^2+1}}$$

First $\cdot$ d(Second) + d(First) $\cdot$ Second

A *differential equation* is an equation involving an unknown function and one or more of its derivatives. Show that the function $y = 2e^{-3x}$ is a solution to the differential equation $y' = -3y$

Given - substitute into diff eq

$$(2e^{-3x})' = -3(2e^{-3x}) \ ?$$

$$2 \cdot e^{-3x} \cdot (-3) = -6e^{-3x} \ ?$$

$$-6e^{-3x} = -6e^{-3x} \ \checkmark$$