4.2-Inverse Functions and Their Derivatives

Definitions:

f is a *one-to-one function* if and only if for each output (y) there is a unique input (x) such that $f(x)=y$

Mathematically: Show only solution to $f(a)=f(b)$ is $a=b$

~~Ctr-Ex~~ $f(x)=x^2$

$\sqrt{a^2}=\sqrt{b^2}$

$a=\pm b$

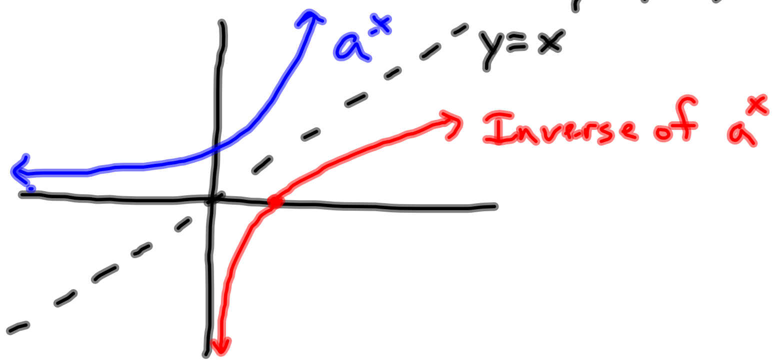
2 answers
$a=b$
$a=-b$
NOT one to one

If f is one-to-one, the *inverse* of f is a function f^{-1} such that if y is in the domain of f^{-1}, then $\boxed{\begin{gathered}x=f^{-1}(y)\\ y=f(x)\end{gathered}}$ means

Alt
$y=f^{-1}(x)$ means
$x=f(y)$

If (a, b) is on the graph of $y = f(x)$, then (b, a) is on the graph of $y = f^{-1}(x)$

Therefore, the graph of $y = f^{-1}(x)$ is a reflection of $y = f(x)$ about $y = x$.



If f is one-to-one and differentiable at $x = g(a)$ and $g = f^{-1}$, then

$$* \boxed{g'(a) = \frac{1}{f'(g(a))}} *$$

$y = g(x) = f^{-1}(x)$ means

$x = f(y)$ Implicit Diff

$$1 = f'(y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

$$\frac{dy}{dx} = \frac{1}{f'(g(x))}$$

Examples:

Find the inverse of $f(x) = \dfrac{2x-4}{x+3}$

Step 1: Switch x and y

$$x = \frac{2y-4}{y+3}$$

Step 2: Solve for y

$$x(y+3) = 2y-4$$

$$xy + 3x = 2y - 4$$

$$xy - 2y = -3x - 4$$

$$y(x-2) = -3x - 4$$

$$\boxed{y = \frac{-3x-4}{x-2} = f^{-1}(x)}$$

First, I am going to show (YOU DON'T HAVE TO!) that f is one to one

$$f(a) = f(b)$$

$$\frac{2a-4}{a+3} = \frac{2b-4}{b+3} \quad \text{Solve for } a$$

$$(2a-4)(b+3) = (2b-4)(a+3)$$

$$\cancel{2ab} + 6a - 4b - \cancel{12} = \cancel{2ab} + 6b - 4a - \cancel{12}$$

$$+4a \quad +4b \qquad\qquad +4b \quad +4a$$

$$10a = 10b$$

$$a = b \ \checkmark$$

observations:

	f	f^{-1}
y-int	$(0, -\frac{4}{3})$	$(0, 2)$
x-int	$(2, 0)$	$(-\frac{4}{3}, 0)$
V Asym	$x = -3$	$x = 2$
H Asym	$y = 2$	$y = -3$

WARNING: Find inverse of $f(x) = \sqrt{x-3}$

$$x = \sqrt{y-3}$$

$$x^2 = y - 3$$

$$y = x^2 + 3$$

Problem: Not one-to-one

Solution: Restrict Domain

$$\boxed{f^{-1}(x) = x^2 + 3 \ ; \ x \geq 0}$$

Mar 9-11:17 AM

Given $g(x)$ is the inverse of $f(x) = x + 2\sqrt{x}$, find $g'(8)$

Formula: $g'(a) = \dfrac{1}{f'(g(a))}$ $\quad a=8$

So $g'(8) = \dfrac{1}{f'(g(8))}$

$f'(x) = 1 + 2 \cdot \frac{1}{2} x^{-1/2}$

$= 1 + \frac{1}{\sqrt{x}}$

? What do we substitute in for x?

$g(8)$?

$g(8) = y$ means

$f^{-1}(8) = y$ means

$f(y) = 8$

$y + 2\sqrt{y} = 8$ By inspection

$y = 4$ This is $g(8)$

$g'(8) = \dfrac{1}{f'(4)}$

$= \dfrac{1}{1 + \frac{1}{\sqrt{4}}}$

$= \dfrac{1}{1 + \frac{1}{2}} \quad \frac{3}{2}$

$= \boxed{\frac{2}{3}}$

m_{tan} to $g(x)$ at $(8,4)$ is $\frac{2}{3}$

m_{tan} to $f(x)$ at $(4,8)$ is $\frac{3}{2}$