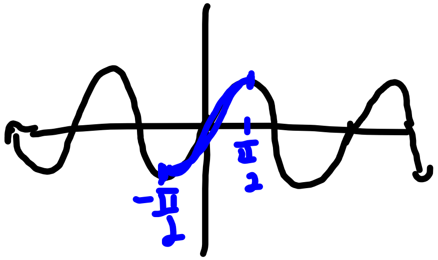
4.6-Inverse Trig Functions and their Derivatives

sin, cos tan and one-to-one functions:

Not one-to-one



Restrict Domain

$y = \sin x$; $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is one-to-one

What angle?

$y = \sin^{-1} x$ (or $\arcsin x$) if and only if $x = \sin y$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \cos^{-1} x$ if and only if $x = \cos y$ and $y \in [0, \pi]$

$y = \tan^{-1} x$ if and only if $x = \tan y$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Derivative of $\sin^{-1}$:

$y = \sin^{-1} x$ means

$x = \sin y$ (Implicit Diff)

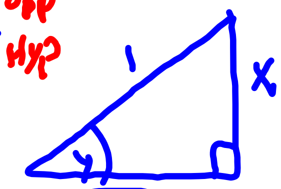
$1 = (\cos y)\, y'$

$y' = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$? simplify

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}}$$

Reference Triangle

$\sin y = \frac{x}{1}$ $\frac{\text{Opp}}{\text{Hyp}}$



$\sqrt{1-x^2}$ by Pythagoras

$\cos y = \frac{\sqrt{1-x^2}}{1}$ $\frac{\text{Adj}}{\text{Hyp}}$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1}x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1}x\right) = \frac{-1}{x\sqrt{x^2-1}}$$

Examples:

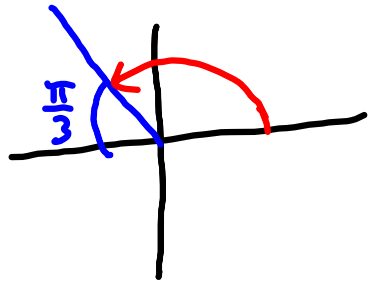
Find the exact value of $\cos^{-1}\left(-\frac{1}{2}\right)$

(Thought process):

$y = \cos^{-1}\left(-\frac{1}{2}\right)$ means

$\cos y = -\frac{1}{2}$ $\qquad \cos\frac{\pi}{3} = \frac{1}{2}$

Quad II (cosine is negative)

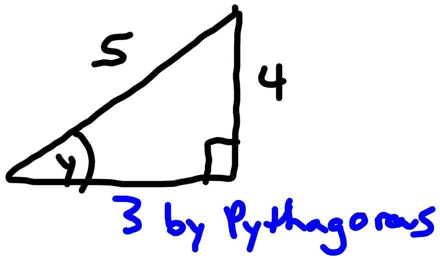


$$\boxed{y = \frac{2\pi}{3}}$$

Compute $\cot(\underbrace{\sin^{-1}\left(\frac{4}{5}\right)}_{y})$

$y = \sin^{-1}\frac{4}{5}$ means

$\sin y = \frac{4}{5}$



$$\cot y = \frac{\cos y}{\sin y} \quad \frac{3/5}{4/5} = \boxed{\frac{3}{4}}$$

Side Note:

$$\cancel{\sin\left(2\sin^{-1}\left(\frac{4}{5}\right)\right) = \frac{8}{5}}$$

$$\sin\left(2\underbrace{\sin^{-1}\left(\frac{4}{5}\right)}_{y}\right)$$

$$= \sin(2y) = 2\sin y \cos y$$

use triangle

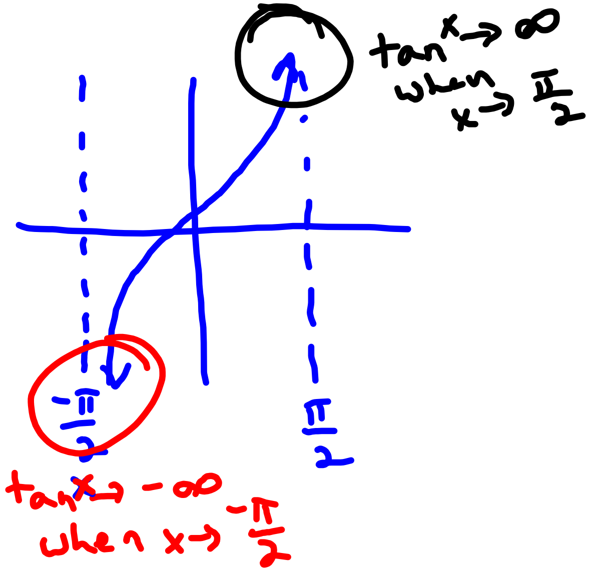
Compute $\lim_{x\to 0} \tan^{-1}\left(\frac{1-x}{2x^2}\right)$

Since $\tan^{-1}$ is cts,

$$\lim_{x\to 0} \frac{1-x}{2x^2} \quad \frac{1}{0} \quad \frac{+}{+} = +\infty$$

$$= \text{"} \tan^{-1}(\infty) \text{"}$$

$$= \frac{\pi}{2}$$



Find the derivative of $f(x) = \arcsin\left(\frac{1}{x}\right)$ (and simplify)

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \left(\frac{-1}{x^2}\right)$$

$$= \frac{-1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{-1}{\frac{x^2}{x}\sqrt{\left(1-\frac{1}{x^2}\right)x^2}}$$

$\sqrt{x^2} = x$

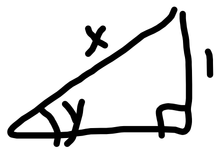
$$= \boxed{\frac{-1}{x\sqrt{x^2-1}}} = \frac{d}{dx}\left(\csc^{-1}x\right)$$

NOTE

$y = \sin^{-1}\left(\frac{1}{x}\right)$ $\qquad$ $y = \csc^{-1}x$

$\sin y = \frac{1}{x}$ $\qquad$ $\csc y = \frac{x}{1}$

$\sin y = \frac{1}{x}$



NOTE: $\csc^{-1}x \neq \frac{1}{\sin^{-1}x}$

Trig Identities do NOT apply to Inv Trig functions!