

4.8 L'Hospital's Rule

Goal: Given a limit of indeterminate form ($0/0$, ∞/∞ , etc) with differentiable functions, find the limit.

L'Hospital's Rule: If f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a), and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ or $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \quad \text{NOT quotient rule!}$$

Examples:

Find each of the following limits:

$$\lim_{t \rightarrow -2} \frac{t^3 - t^2 - t + 10}{t^2 + 3t + 2} = \frac{0}{0}$$

$$= \lim_{t \rightarrow -2} \frac{3t^2 - 2t - 1}{2t + 3} = \frac{12 + 4 - 1}{-4 + 3} = \boxed{-15}$$

Bonus Example

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{-\sin x}{2x} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{-\cos x}{2} = \boxed{-\frac{1}{2}}$$

Approaching zero, not = 0

$$\lim_{x \rightarrow 0^+} x \ln x$$

Must have fraction to use L'Hospital's

$$= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow 0^+} -x = 0$$

Dominance among exp/power/log functions:

$$\textcircled{1} \quad \lim_{x \rightarrow \infty} \frac{\ln x}{x^p} \quad \frac{\infty}{\infty} \quad (p > 0)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{p x^{p-1}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{p x^p} = 0$$

$\ln x \rightarrow \infty$ slower than any (positive) power of x .

$$\textcircled{2} \quad \lim_{x \rightarrow \infty} \frac{e^x}{x^p} \quad \frac{\infty}{\infty} \quad (p > 0)$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{p x^{p-1}} \quad \begin{array}{l} \text{if } p-1 \leq 0 \quad \frac{\infty}{\#} \rightarrow \infty \\ \text{if } p-1 > 0 \quad \frac{\infty}{\infty} \text{ repeat L'Hospital's} \end{array}$$

$$= \lim_{x \rightarrow \infty} \frac{e^x}{p(p-1)x^{p-2}}$$

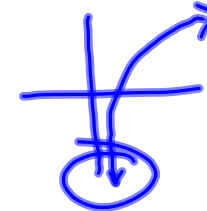
No matter how large p is, eventually exponent ≤ 0 , so we'll have $\frac{\infty}{\#} \rightarrow \infty$

$e^x \rightarrow \infty$ faster than any power of x .

$$\lim_{x \rightarrow 0} x^{\sin x}$$

Apply ln to function

Look at $\lim_{x \rightarrow 0} \ln(x^{\sin x}) = \lim_{x \rightarrow 0^+} \sin x \cdot \ln x$



If you start by bring in ln, end by apply exp!

$$= e^{\lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x}}$$

$$= e^{\lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}}$$

-sin x tan x

$$= e^{\lim_{x \rightarrow 0^+} \frac{-\sin x \tan x}{x}}$$

$$= e^0 = 1$$

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$