4.8 L'Hospital's Rule

Goal: Given a limit of indeterminate form (0/0, ∞/∞, etc) with differentiable functions, find the limit.

L'Hospital's Rule: If f and g are differentiable and $g'(x) \neq 0$ ~~on an open interval I that contains a (except possibly at a)~~, and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

NOT quotient rule!

Examples:

Find each of the following limits:

$$\lim_{t\to -2} \frac{t^3 - t^2 - t + 10}{t^2 + 3t + 2} = \frac{0}{0}$$

$$= \lim_{t\to -2} \frac{3t^2 - 2t - 1}{2t + 3} = \frac{12 + 4 - 1}{-4 + 3} = \boxed{-15}$$

Bonus Example

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} \quad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-\sin x}{2x} \quad \frac{0}{0} \quad 1$$

$$= \lim_{x\to 0} \frac{-\cos x}{2} = \boxed{-\frac{1}{2}}$$

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$0 \cdot -\infty$ Approaching zero, not $= 0$

$$\lim_{x\to 0^+} x \ln x$$

Must have fraction to use L'Hospitals

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \quad \frac{-\infty}{\infty}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\cancel{x}}}{-\frac{1}{x^2}} \quad -\frac{x^{\cancel{2}}}{1}$$

$$= \lim_{x\to 0^+} -x = \boxed{0}$$

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Dominance among exp/power/log functions:

① $\lim_{x\to\infty} \frac{\ln x}{x^p} \quad \frac{\infty}{\infty} \quad (p>0)$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{p x^{p-1}} \quad \frac{1}{p x^{p-1}}$

$= \lim_{x\to\infty} \frac{1}{p x^{p}_{\infty}} = 0$

$\boxed{\text{ln } x \to \infty \text{ \underline{slower} than any (positive) power of } x.}$

② $\lim_{x\to\infty} \frac{e^x}{x^p} \quad \frac{\infty}{\infty} \quad (p>0)$

$= \lim_{x\to\infty} \frac{e^x}{p x^{p-1}}$ if $p-1 \le 0$ $\frac{\infty}{\#} \to \infty$

if $p-1>0$ $\frac{\infty}{\infty}$ repeat L'Hospital's

$= \lim_{x\to\infty} \frac{e^x}{p(p-1)x^{p-2}}$

No matter how large p is, eventually exponent ≤ 0, so we'll have $\frac{\infty}{\#} \to \infty$

$\boxed{e^x \to \infty \text{ \underline{faster} than any power of } x.}$

$\lim_{x \to 0} x^{\sin x}$ 0^0

Apply ln to function

Look at $\lim_{x\to 0} \ln(x^{\sin x}) = \lim_{x\to 0^+} \sin x \cdot \ln x$

$e^{\lim_{x\to 0} \ln(x^{\sin x})} = e^{\lim_{x\to 0^+} \sin x \cdot \ln x}$

$= e^{\lim_{x\to 0^+} \frac{\ln x}{\csc x}}$ $\frac{-\infty}{\infty}$

$= e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}}$ $-\sin x \tan x$

$= e^{\lim_{x\to 0^+} \frac{-\sin x \tan x}{x}}$ $\frac{\sin x}{x} \to 1$, $\tan x \to 0$

$= e^0 = \boxed{1}$

* If you start by bring in ln, end by apply exp! *

$\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$