1 2.5: Continuity

Definitions:

$f$ is continuous at $x = a$ if and only if \[ \lim_{x \to a} f(x) = f(a) \]

(3 implications)

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. Are they equal? (limit = y-value)

Removable Discontinuities

A discontinuous function $f$ has a removable discontinuity at $x = a$ if and only if there is a function $g$ such that $g$ is continuous and $g(x) = f(x)$ for all $x \neq a$. \[ \lim_{x \to a} f(x) \text{ exists} \]
Source for understanding: Maplet “Left and Right Hand Limits and Continuities”, located at http://calc1ab.math.tamu.edu/maple/maplets/ (NetID login)
Theorems:

Limits inside Continuous Functions

If \( f \) is continuous,

\[
\lim_{x \to a} f(g(x)) = f\left( \lim_{x \to a} g(x) \right)
\]

Continuity of Polynomial/Rational Functions

Continuous everywhere on their domains

Intermediate Value Theorem

If \( f \) is continuous on \([a, b]\) and \( N \) is between \( f(a) \) and \( f(b) \), then there is a \( c \in [a, b] \) such that \( f(c) = N \)
Examples:

If \( f(x) = \begin{cases} 
1 - x & \text{if } x \geq 1 \\
-x & \text{if } x < 1 
\end{cases} \)

determine whether \( f \) is continuous at \( x = 1 \) or not. Explain your answer precisely. Is \( f \) continuous from the left or right? Does \( f \) have a removable discontinuity?

\[
\begin{align*}
\lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (1 - x) = 0 \\
\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (-x) = -1 \\
f(1) &= 0 \\
\lim_{x \to 1} f(x) &= \text{DNE (left \neq right)}
\end{align*}
\]

\( \therefore f \) not continuous \( (\text{limit DNE}) \)

Left? \( \text{NO (limit \neq y-value)} \)
Right? \( \text{YES (limit = y-value)} \)
Removable? \( \text{NO (limit DNE)} \)
\[
\lim_{x \to 1} \sqrt{\frac{x^2 + 2x - 3}{x - 1}} = \sqrt{\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}}
\]

If \( f \) is continuous, \( \lim_{x \to a} f(g(x)) = f(\lim_{x \to g(x)} g(x)) \)

\[
= \sqrt{\lim_{x \to 1} \frac{(x+1)(x-1)}{x-1}}
\]

\[
= \sqrt{\lim_{x \to 1} (x+1)}
= \sqrt{2}
\]
IVT  If \( f \) is cts on \([a, b]\) and \( N \) is between \( f(a) \) and \( f(b) \),
then there is a \( c \) such that \( f(c) = N \) (i.e. there is a solution
to \( f(x) = N \)).

Is there a real solution to the equation \( x^5 - x^2 + 2x = 6 \)? If so, find the value of \( a \) such there is a
solution in the interval \([a, \, a + 1]\).

\[
\begin{align*}
a &= \phantom{0} \\
b &= \\
\text{Choose} \quad f(a) &= x^5 - x^2 + 2x \\N &= 6 \\
\text{Also can say} \quad f(b) &= x^5 - x^2 + 2x - 6 \\N &= 0 \\
\end{align*}
\]

\( N = 0 \)

\( \begin{align*}
\text{Let } a &= 0 \quad f(b) = 0^5 - 0^2 + 2 \cdot 0 = 0 \\
b &= 1 \quad f(1) = 1^5 - 1^2 + 2 \cdot 1 = 2 \\
2b &= f(b) = 2^5 - 2^2 + 2 \cdot 2 = 32
\end{align*} \)

If \( f \) is cts on \([a, b]\) (\( f \) is a polynomial) and \( N \) is between \( f(a) \) and \( f(b) \) (\( 0 < b < 32 \)) then by IVT there is a solution to \( x^5 - x^2 + 2x = 6 \),
between \( 0 \) and \( 2 \).

Find \( a \)?

\( f(1) = 2 \) so \( a = 1 \)

solution between 1 and 2

\( \begin{align*}
\text{Bisection Method} \\
1 \text{ and } 1.5 \\
1 \text{ and } 1.25
\end{align*} \)
On Your Own: Determine whether the function \( f(x) = \begin{cases} \frac{x^2 - 2x - 8}{x + 2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases} \)

is continuous at \( x = 2 \) or not. Explain your answer precisely. Is \( f \) continuous from the left or right? Does \( f \) have a removable discontinuity?

\[
\sqrt{f(2) = 3} \\
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 2x - 8}{x + 2} = \lim_{x \to 2} \frac{(x-4)(x+2)}{x+2} = -2 - 4 = -6 \\
\text{(No need for left/right since same equation)}
\]

\[
\lim_{x \to 2} f(x) \neq f(2) \quad \therefore \text{Not cts} \\
\text{Left Limit} = -6 \neq 3 \quad \text{Not cts left} \\
\text{Right Limit} = -6 \neq 3 \quad \text{Not cts right} \\
\lim \text{Exists } \rightarrow f \text{ has a removable discontinuity.}