1 2.7: Tangents, Velocities, and Rates of Change

We are now ready to find a formal way of computing the slope of the line tangent to the curve. Re-view the animation from 2.1 posted on my webpage. What happens as the second x-coordinate moves closer to the given tangent line point?

\[ y = x^2 \]

In this picture:

\[ m_{sec} = \frac{\Delta y}{\Delta x} = \frac{(0.1)^2 - (1)^2}{1.3 - 1} \]

\[ = \frac{0.1}{0.3} = 2.3 \]
Each secant line above passes through the point (1,1). If \( x \) is the \( x \)-coordinate of the second point, write an expression for the slope of the line between the two points.

\[
m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{x^2-1}{x-1}
\]

Write and solve a limit problem which allows us to find the slope of the tangent line at \( x = 1 \).

\[
m_{\text{tan}} = \lim_{x \to 1} \frac{x^2-1}{x-1}
\]

\[
= \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}
\]

\[
= \lim_{x \to 1} (x+1)
\]

\[
= 2
\]
More General: Draw any function, a tangent line, and a secant line. Label the $x$-coordinate of the tangent line point $a$ and the $x$-coordinate of the second point $x$.

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x-a}$$

What should happen as the point $(x, f(x))$ moves closer to the tangent-line point $(a, f(a))$? Write a limit which explains this mathematically:

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$
**Most General:** Draw any function, a tangent line, and a secant line again in the space below. Label the $x$-coordinate of the tangent line $a$ and let $h$ be the horizontal distance between the $x$ values on the secant line. (View the animation posted under today’s notes for a visual understanding of this.)

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
The derivative of a function at \( x = a \) is given by

\[
\frac{df}{dx} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

Examples:

Use a limit definition to find the equation of the line tangent to the curve \( f(x) = \frac{8}{x+2} \) at the point where \( x = 0 \).

Specific point

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}
\]

Can use \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) or \( \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \)

Easier algebraically in most cases

\[
\lim_{x \to 0} \frac{\frac{8}{x+2} - 4}{x}
\]

\[
= \lim_{x \to 0} \frac{\frac{8}{x+2} - \frac{4(x+2)}{x+2}}{x}
\]

\[
= \lim_{x \to 0} \left( \frac{8 - 4(x+2)}{x+2} \div 1 (x+2) \right)
\]

\[
= \lim_{x \to 0} \left( \frac{-4x}{x+2} \right)
\]

\[
= \lim_{x \to 0} \frac{-4}{x+2} = -2 \text{ slope}
\]

Equation: need point + slope

\[
m = -2
\]

Point \( x = 0, y = \frac{8}{0+2} = 4 \)

\((0, 4)\)

Equation

\[
y = -2x + 4
\]
**On Your Own:** Find the equation of the line tangent to the curve $f(x) = x^2 - 4x + 4$ at the point where $x = 3$.

\[
\text{Slope } f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{(x^2 - 4x + 4) - (3^2 - 4 \cdot 3 + 4)}{x - 3} = \lim_{x \to 3} \frac{x^2 - 4x + 3}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x - 1)}{x - 3} = 2
\]

**Point**

\[
f(3) = \begin{cases} 
  y - 1 &= 2(x - 3) \text{ or } \\
  y &= 2x - 5
\end{cases}
\]
Secant and Tangent Vectors—an Introduction

Given \( \vec{r}(t) \), find a vector tangent to the graph at a given point.

Secant vector:
\[
\vec{r}(t) = \vec{r}(a) + \text{average position over time} \quad \text{t = time}
\]

Tangent vector:
\[
\lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}
\]

Alternate:
\[
\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}
\]

Instantaneous velocity:
\[
\lim_{t \to a} \frac{\vec{r}(t) - \vec{r}(a)}{t-a}
\]
Find a vector tangent to the curve \( r(t) = \sqrt{t - 11} + (t^2 + t - 2)j \) at the point \((1,4)\). Then find parametric equations of the line tangent to the curve at this point.

\[
\lim_{t \to a} \frac{\vec{r}(t) - \vec{r}(a)}{t - a} \quad \text{or} \quad \lim_{h \to 0} \frac{\vec{r}(a+h) - \vec{r}(a)}{h}
\]

What is \( a \)?

\[
\sqrt{t - 1} = 1 \quad t^2 + t - 2 = 4
\]

\[
t = 2\quad t = 2
\]

\[
\lim_{t \to 2} \frac{\vec{r}(t) - \vec{r}(2)}{t - 2}
\]

\[
= \lim_{t \to 2} \frac{[\sqrt{(t-1)}i + (t^2 + t - 2)j] - [1i + 4j]}{t - 2}
\]

\[
= \lim_{t \to 2} \frac{((\sqrt{t-1} - 1)i + (t^2 + t - 2 - 4)j)}{t - 2}
\]

\[
= \lim_{t \to 2} \frac{(\sqrt{t-1} - 1)i + (t^2 + t - 6)j}{t - 2}
\]

\[
= \left( \lim_{t \to 2} \frac{\sqrt{t-1} - 1}{t - 2} \right)i + \left( \lim_{t \to 2} \frac{t^2 + t - 6}{t - 2} \right)j
\]

\[
= \left( \lim_{t \to 2} \frac{t - 1}{(t-1)(t+1)} \right)i + \left( \lim_{t \to 2} \frac{t^2 + t - 6}{t - 2} \right)j
\]

\[
= \left( \frac{1}{2} \right)i + \left( \frac{5}{2} \right)j
\]

Parametric Equation of tangent line:

\[
\vec{r}_0 + t\vec{v} \quad \text{(vector)}
\]

\[
\vec{r}_0 = 1i + 4j
\]

\[
\vec{v} = \frac{1}{2}i + 5j
\]

\[
\vec{r}(t)=(1i + 4j) + t\left( \frac{1}{2}i + 5j \right)
\]

\[
= (1 + \frac{1}{2}t)i + (4 + 5t)j
\]

\[
\text{or}
\]

\[
x = 1 + \frac{1}{2}t
\]

\[
y = 4 + 5t
\]