1 3.1: The Derivative

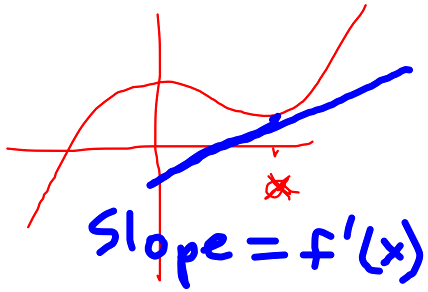
Now that we can find the slope of the line tangent to a curve at any point (provided the limit of the slope exists), we can talk about a new function based on this calculation.

Definition: The **derivative function** of a function f (or the **derivative** of f) is a function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Input : x-coord

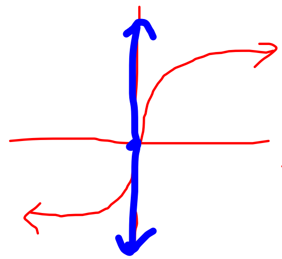
Output : slope of tan line

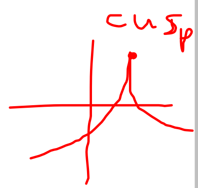


When is f not **differentiable**? (i.e., when does $f'(x)$ not exist or when is x not in the domain of f'?

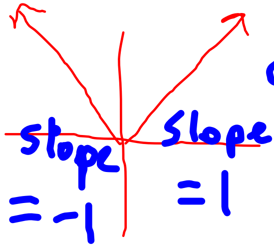
1) Infinite limit $\frac{\#}{0}$ (slopes infinite)

Vertical Tangent



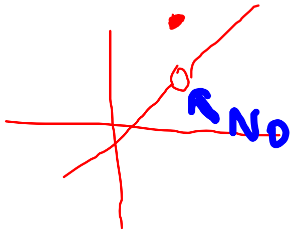


2) Left/Right $\neq$



corner

3) Not cts



No tangent line

Examples:

Let $f(x) = \dfrac{8}{x+2}$. Find $f'(x)$ and use it to determine the slope of the line tangent to f at the point where $x = 0$, $x = 2$ and $x = -1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{8}{x+h+2} - \frac{8}{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x+2}{x+2}\cdot\frac{8}{x+h+2} - \frac{8}{x+2}\cdot\frac{x+h+2}{x+h+2}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{8x+16 + -(8x + 8h + 16)}{(x+2)(x+h+2)}\right)$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\frac{\cancel{8x}+\cancel{16}-\cancel{8x}-8\cancel{h}-\cancel{16}}{(x+2)(x+h+2)}\right)$$

$$f'(x) = \lim_{h\to 0} \frac{-8}{(x+2)(x+\underset{\to 0}{h}+2)} = \boxed{\frac{-8}{(x+2)^2}}$$

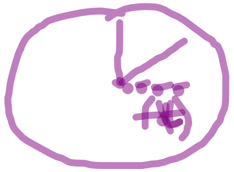
$$f'(x) = \frac{-8}{(x+2)^2}$$

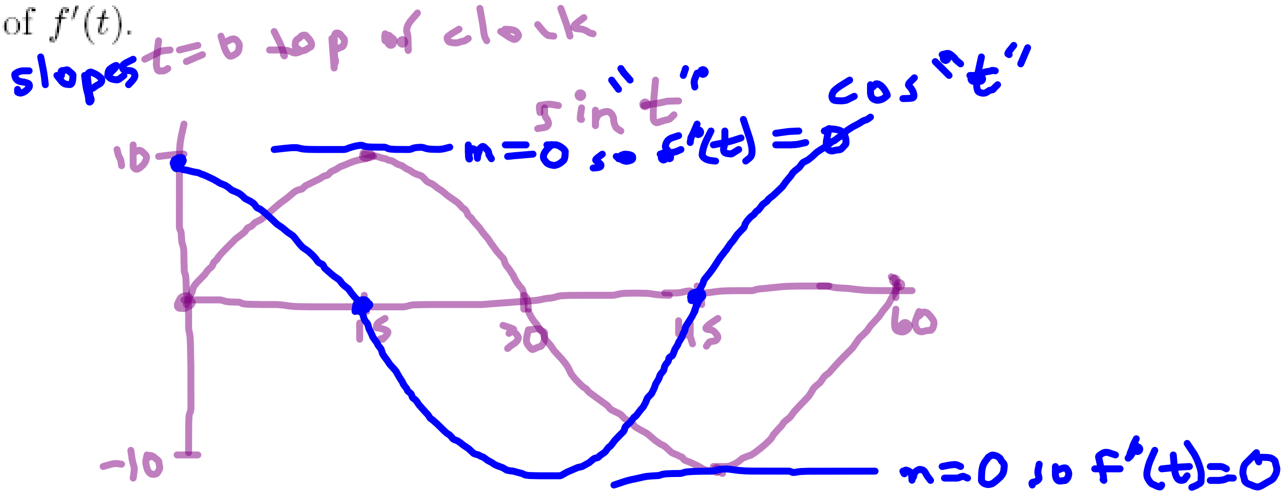
$x=0$: $m = f'(0) = \dfrac{-8}{(0+2)^2} = \boxed{-2}$

$x=2$: $m = f'(2) = \dfrac{-8}{(2+2)^2} = \boxed{-\frac{1}{2}}$

$x=-1$: $m = f'(-1) = \dfrac{-8}{(-1+2)^2} = \boxed{-8}$

A clock has a radius of 10 cm. Let $f(t)$ be the horizontal position of the tip of the second hand (where $f(t) = 0$ refers to the diameter through the 12 and 6). Sketch a rough graph of $f(t)$, then sketch the graph of $f'(t)$.





On Your Own: Determine whether $f(x) = |x^2 - 9|$ is differentiable at $x = 3$.

$$f(x) = |x^2-9| = \begin{cases} x^2-9 & \text{if } x^2-9 \geq 0 \quad (x \geq 3 \text{ or } x \leq -3) \\ -(x^2-9) & \text{if } x^2-9 \leq 0 \quad (-3 < x < 3) \end{cases}$$

$+ \quad - \quad +$; -3, 3

$$= \begin{cases} x^2-9 & \text{if } x \leq -3 \\ -x^2+9 & \text{if } -3 < x < 3 \\ x^2-9 & \text{if } x \geq 3 \end{cases}$$

(at one point)

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x-3}$$

Left

$$= \lim_{x \to 3^-} \frac{-x^2+9-0}{x-3}$$

$$= \lim_{x \to 3^-} \frac{-(x+3)\cancel{(x-3)}}{\cancel{x-3}}$$

$$= -6$$

Right

$$\lim_{x \to 3^+} \frac{x^2-9-0}{x-3}$$

$$= \lim_{x \to 3^+} \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}}$$

$$= 6$$

Left & Right $\neq$ $\therefore$ $f'(3)$ DNE

(Graph)