1 4.2: Inverse Functions

functions vs. one-to-one functions:

Recall: Function

Each input \( \rightarrow \) at most one output

One-to-One Function

Each output also \( \rightarrow \) at most one input

Show one-to-one?

1) if graph, pass the horizontal line test
2) ONLY solution to \( f(a) = f(b) \) is \( a = b \).

If \( f \) is one-to-one, the inverse of \( f \) is a function \( f^{-1} \) such that

\[ y = f^{-1}(x) \text{ if and only if } x = f(y) \] (\( x \)'s and \( y \)'s switch places)
If \((a, b)\) is on the graph of \(y = f(x)\), then \((b, a)\) is on the

graph of \(y = f^{-1}(x)\).

Result: Graph of \(y = f^{-1}(x)\) is a reflection of
\(y = f(x)\) about the line \(y = x\)

If \(f\) is one-to-one and differentiable at \(x = g(a)\), where \(g = f^{-1}\), then

\[
\begin{align*}
g'(x) &= \frac{1}{f'(g(x))} \\
g'(a) &= \frac{1}{f'(g(a))}
\end{align*}
\]

\begin{itemize}
  \item \text{Iden:}
  \item \(y = g(x)\) means \(x = f(y)\) \hspace{1cm} \text{Implicit Diff}
  \item \(1 = f'(y) y'\)
  \item \(y' = \frac{1}{f'(y)} = \frac{1}{f'(g(x))}\)
\end{itemize}
Examples:

Show \( f(x) = \frac{2-x}{2+x} \) is one-to-one and find \( f^{-1} \).

\[
\begin{align*}
  f(a) &= f(b) \\
  \frac{2-a}{2+a} &= \frac{2-b}{2+b} \\
  (2-a)(2+b) &= (2-b)(2+a) \\
  4+2b-2a-2b &= 4+2a-2b-2a \\
  -4a &= -4b \\
  a &= b \text{ only solution} \\
  \therefore f \text{ is one-to-one}
\end{align*}
\]
TRUE OR FALSE? The inverse of $f(x) = \sqrt{x - 1}$ is $g(x) = x^2 + 1$. Explain.

\[ y = \sqrt{x - 1} \]
\[ x = \sqrt{y - 1} \]
\[ x^2 = y - 1 \]
\[ y = x^2 + 1 \]

\[ g(x) = x^2 + 1; \quad x \geq 0 \]

\[ \therefore \text{if } f(x) = \sqrt{x - 1}, \]
\[ f^{-1}(x) = x^2 + 1; \quad x \geq 0 \]

\[ \text{parabola - NOT one-to-one} \]
\[ \text{Restrict domain of } g(x) \]
\[ \text{IS one-to-one} \]
Given $g$ is the inverse of $f(x) = x^5 - x^3 + 4x$, find $g'(4)$.

$$g'(4) = \frac{1}{f'(g(4))}$$

$$= \frac{1}{f'(1)}$$

$$= \frac{1}{5(1)^4 - 3(1)^3 + 4y} = \frac{1}{6}$$

$$f'(x) = 5x^4 - 3x^3 + 4$$

$$g(4) = y$$

$f(y) = 4$

$y^5 - y^3 + 4y = 4$

$y = 1 \quad \text{by inspection}$
(On your own): The function $f(x) = \tan x$ is one-to-one on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $g = f^{-1}$, find $g'(1)$.

\[
g'(1) = \frac{1}{f'(g(1))}
\]

\[
= \frac{1}{\sec^2 \left(\frac{\pi}{4}\right)}
\]

\[
= \cos^2 \left(\frac{\pi}{4}\right)
\]

\[
= \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}
\]

$f'(x) = \sec^2 x$

$g(1) = y$ means

$f(y) = 1$

$\tan y = 1$

$y = \frac{\pi}{4}$