1 4.5: Exponential Growth and Decay

A solution to the differential equation $y' = ky$ is:

$y(t) = Ce^{kt}$

(C, k constants)

Check: $\text{LHS } y' = Ce^{kt} \cdot k$

RHS $ky = k \cdot Ce^{kt}$

$\sqrt{y}' = ky$
Exponential Growth and Decay:

"The rate of change in a quantity is proportional to the amount of the quantity present" (Let $y =$ quantity)

\[
\frac{y'}{y} = k \quad \rightarrow \quad y' = ky
\]

So \( y = C e^{kt} \)

**Goal:** Use given information to find $C$ and $k$.

- $C \rightarrow$ initial data
- $k \rightarrow$ subsequent reading
Examples:

Agigium is a radioactive substance with a half-life of 105 days. If there are 2013g of Agigium initially, how much remains after $t$ days? How much remains after 200 days? When will there be only 100g left?

$$y = Ce^{kt} \quad t = \text{time (days)}$$

$y = (g)$ of Agigium

$$2013 = Ce^{k \cdot 0} \quad \rightarrow \quad C = 2013$$

$$y = 2013e^{kt}$$

$$\frac{2013}{2} = 2013e^{k(105)}$$

$$\ln \frac{1}{2} = k \cdot 105$$

$$105k = \ln \left(\frac{1}{2}\right)$$

$$k = \frac{1}{105} \cdot \ln \left(\frac{1}{2}\right)$$

$\text{Simplified algebraically:}$

$$y = 2013 e^{\frac{1}{105} \ln \left(\frac{1}{2}\right) t}$$

$$= 2013 e^{\ln \left(\frac{1}{2}\right) \cdot \frac{t}{105}}$$

$$= 2013 \left(\frac{1}{2}\right)^{\frac{t}{105}}$$

$c) \quad y = 100 \quad \text{Solve for } t$
A 100-gallon drum contains 5 lbs of salt. Pure water enters the drum at a rate of 10 gallons per minute. The solution is thoroughly mixed and leaves the drum at the same rate. Find an equation which gives the amount of salt in the drum at any time \( t \).

Let \( y \) = amount of salt in tank (lbs)

\[
\frac{dy}{dt} = \text{rate in} - \text{rate out}
\]

\[
y' = 0 - \left(\frac{y}{10}\right)
\]

\[
y' = -\frac{1}{10}y
\]

\[
y = Ce^{-\frac{1}{10}t}
\]

\[
t=0 \quad y=5
\]

\[
y = Ce^{-\frac{1}{10}t} \rightarrow C=5
\]

\[
y = 5e^{-\frac{1}{10}t}
\]
Newton’s Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference in temperature between the object and its surroundings. A metal ball is brought from 5°C weather into a 20°C room. One minute later, the ball has a temperature of 12°C.

a) Write a function which gives the temperature of the ball at any time \( t \).

Let \( \frac{y'}{y-20} = k \rightarrow y' = k(y-20) \)

\[ u' = ku \]

Solution is \( u = Ce^{kt} \)

\[ y-20 = Ce^{kt} \]

\[ y = 20 + Ce^{kt} \]

\[ t = 0 \quad y = 5 \]

\[ 5 = 20 + Ce^{0} \rightarrow C = -15 \]

\[ y = 20 - 15e^{kt} \]

\[ t = 1, \quad y = 12 \]

\[ 12 = 20 - 15e^{k} \]

\[ -8 = -15e^{k} \]

\[ \frac{8}{15} = e^{k} \]

\[ k = \ln\left(\frac{8}{15}\right) \]

\[ y = 20 - 15e^{\ln\left(\frac{8}{15}\right)t} \]
(On Your Own):

b) What is the temperature one more minute later?

c) When will the ball have a temperature of 18°C?

\[ y = 20 - 15 e^{\frac{\ln(\frac{3}{15})}{t}} \]

\[ y = 20 - 15 e^{\ln(\frac{3}{15})} \cdot 2 \]

\[ = 20 - 15 e^{\ln(\frac{3}{15})^2} \]

\[ = 20 - 15 \cdot \left(\frac{3}{15}\right)^2 \]

\[ = 20 - \frac{64}{15} = \frac{300 - 64}{15} = \frac{236}{15} \approx 15.7^\circ C \]

\[ t = \frac{\ln(\frac{3}{15})}{\ln(\frac{3}{15})} \text{ minutes} \approx 3.2 \text{ minutes} \]