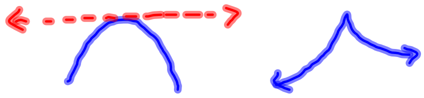
1 5.2: Maxima and Minima

Definitions:

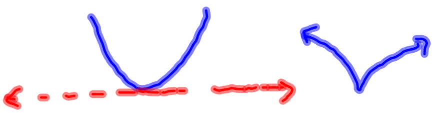
local

f has a **relative maximum at** $x = a$ if and only if $f(x) < f(a)$ for all x "near" $x=a$



local

f has a **relative minimum at** $x = a$ if and only if $f(x) > f(a)$ for all x "near" $x=a$



Fermat's Theorem: If f has a relative maximum or relative minimum at $x = a$ and f is differentiable at $x = a$, then $f'(a) = 0$

More Definitions:

f has a **critical value** at $x = a$ if and only if $f'(a) = 0$ or $f'(a)$ DNE

($\therefore$ we only need to look for relative extrema at critical values

NOTE: critical value $\not\Rightarrow$ relative max or min)

f has an **absolute maximum** at $x = a$ if and only if $f(x) \leq f(a)$ for all x

f has an **absolute minimum** at $x = a$ if and only if $f(x) \geq f(a)$ for all x

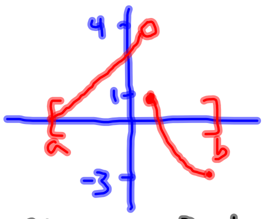
Examples:

$f(x) = \sin x$ Abs max $= 1$
Abs min $= -1$

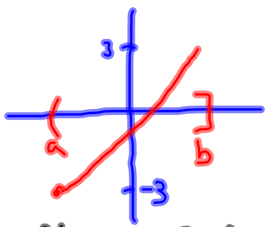
$f(x) = x^2$ No Abs max
Abs min $= 0$

Extreme Value Theorem If f is continuous on a closed, bounded interval, then f attains its absolute maximum and absolute minimum on that interval.

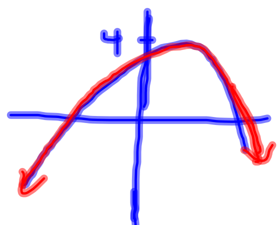
Graphical examples to show that each of the conditions must hold to guarantee the conclusion:



Abs min = -3 when x=b
No abs max



Abs max=3 when x=b
No abs min



Abs max= 4 at crit val
No abs min

NOTE: Abs max/min occur at critical values or endpoints

Process to find:
1) Find critical values and endpoints
2) Evaluate f(#)

Examples:

Find the absolute maximum and absolute minimum of $f(x) = \dfrac{\ln x}{x^2}$ on the interval $(0, 3)$.

Find critical values:

$$f'(x) = \frac{x^2 \cdot \frac{1}{x} - \ln x\,(2x)}{x^4} = 0$$

Crit Value at $x = 0$ Not in interval

$$x - 2x \ln x = 0$$

$$x(1 - 2\ln x) = 0$$

$\cancel{x = 0}$

$\ln x = \frac{1}{2}$

$x = e^{1/2}$

f: ↗ ↘

f': $+$ | $e^{1/2}$ | $-$

Test $x = 1$

Test $x = e$

$$\frac{e - 2e(\ln e)}{e^4}$$

$$\boxed{\begin{array}{l} \text{Abs max} = \dfrac{\ln e^{1/2}}{(e^{1/2})^2} = \dfrac{1}{2e} \text{ when } x = e^{1/2} \\ \text{No abs min} \end{array}}$$

A metal pipe 120cm long is to be cut and used to make a rectangular-shaped "U bar" to be attached to a wall (see figure below). What dimensions will maximize the area enclosed by the U-bar? (Ignore the size of connecting pieces)

Goal: Max $A = lw$

Restriction: $120 = 2w + l$

$l = 120 - 2w \geq 0$

Max $A = (120-2w)w$

$A = 120w - 2w^2; \quad w \in [0, 60]$

crit vals

$A' = 120 - 4w = 0$

$w = 30$

$A(0) = 0$

$A(30) = (120 - 2 \cdot 30) \cdot 30 = 1800$

$A(60) = 0$

$\boxed{\text{Max dimensions are } 30 \times 60 \text{ cm}}$

Nov 11-12:33 PM

On Your Own:

Find the absolute maximum and absolute minimum of $f(x) = 4x^3 - 15x^2 + 12x + 7$ on the interval $0 \le x \le 3$.

crit vals: $f'(x) = 12x^2 - 30x + 12 = 0$

$6(2x^2 - 5x + 2) = 0$

$6(2x-1)(x-2) = 0$

$x = \frac{1}{2}$ $\quad x = 2$ $\qquad$ endpoints $0, 3$

$f(0) = 7$

$f(\frac{1}{2}) = 4(\frac{1}{2})^3 - 15(\frac{1}{2})^2 + 12(\frac{1}{2}) + 7$

$= \frac{1}{2} - \frac{15}{4} + 6 + 7 = \frac{39}{4}$

$f(2) = 4(2)^3 - 15(2)^2 + 12(2) + 7$

$= 32 - 60 + 24 + 7 = \boxed{3 \text{ Abs min}}$

$f(3) = 4(3)^3 - 15(3)^2 + 12(3) + 7$

$= 108 - 135 + 36 + 7 = \boxed{16 \text{ Abs max}}$