1 5.5: Applied Max/Min Problems

**Goal:** To optimize a practical value subject to certain restrictions (often given geometrically)

**Examples:**

A closed rectangular storage container with a square base is to have a volume of 2250 cm$^3$. Material for the top and bottom cost $0.20 per square centimeter. Material for the sides costs $0.30 per square centimeter. Find the cost of materials for the cheapest such container.

\[
\text{Goal: } \min \quad C = (20)2lw + (30)2lh + (30)2wh
\]

\[
\text{Restrictions: } l = w, \quad V = 2250 = lwh
\]

\[
\text{Goal: } \min \quad C = 0.40l^2 + 0.60lh + 0.60lh = 0.40l^2 + 1.20lh
\]

\[
\text{Restriction } V = 2250 = l^2h \quad \rightarrow \quad h = \frac{2250}{l^2}
\]

\[
\text{Goal: } \min \quad C = 0.40l^2 + 1.20l \left( \frac{2250}{l^2} \right)
\]

\[
C = 0.40l^2 + \frac{2700}{l}; \quad l > 0
\]

\[
C' = (0.80l - \frac{2700}{l^2} = 0) \quad l^2
\]

\[
0.80l^3 - 2700 = 0
\]

\[
l = \sqrt[3]{\frac{2700}{0.8}} = \sqrt[3]{\frac{2700}{\frac{8}{10}}} = \frac{30}{2} = 15
\]

**Show min Options:**

1. Closed Bounded Interval method if possible
2. \[ \frac{y}{5} \approx \frac{x}{2} \]
3. Second Derivative Test \( f''(c) > 0 \rightarrow \text{minimum} \)
\( f''(c) < 0 \rightarrow \text{maximum} \)

Min Cost = \((0.40)(15) + \frac{2700}{15} = \boxed{5} \quad \text{Calculator} \)
A poster is to have an area of 150 square inches with 1-inch margins at the sides and bottom and a 2-inch margin at the top. What dimensions will give the largest printed area?

\[
\text{Goal: Max } A_{\text{print}} = (l-2)(w-3) \\
\text{Restriction: } A_{\text{poster}} = 150 = lw
\]

\[
l = \frac{150}{w} = 2
\]

\[
\text{Goal: Max } A = \left(\frac{150}{w} - 2\right)(w-3)
\]

\[
A = 150 - \frac{450}{w} - 2w + 6; \quad w \in [3, 75]
\]

\[
A' = \left(\frac{450}{w^2} - 2 = 0\right)w^3
\]

\[
450 - 2w^2 = 0
\]

\[
450 = 2w^2
\]

\[
225 = w^2
\]

\[
w = 15
\]

\[
\text{Show max } A(15) = 8 \cdot 12 = 96
\]

\[
\text{Max area occurs when } w = 15 \text{ in}, \quad l = \frac{150}{15} = 10 \text{ in}
\]
A wire 40cm long is to be cut into at most 2 pieces. The first piece is bent into a square; the second piece is bent into a circle. How should the wire be divided in order to maximize the total area?

Goal: Max \( A = s^2 + \pi r^2 \)

Restriction: \( 40 = 4s + 2\pi r \)

Solve

\[
\begin{align*}
  s &= \frac{40 - 2\pi r}{4} \\
  10 - \frac{\pi}{2} r &= 0
\end{align*}
\]

Goal: Max \( A = \left(10 - \frac{\pi}{2} r\right)^2 + \pi r^2 \)

\[
\begin{align*}
  A &= 100 - 10\pi r + \frac{\pi^2}{4} r^2 + \pi r^2 \\
  r \in [0, \frac{20}{\pi}]
\end{align*}
\]

\[
\begin{align*}
  A' &= -10\pi + \frac{\pi^2}{2} r + 2\pi r = 0 \\
  (\frac{\pi}{2} r + 2r = 10)^2 \\
  \pi r + 4r &= 20 \\
  r &= \frac{20}{\pi + 4}
\end{align*}
\]

Second Der Test

\[
A'' = \frac{\pi^2}{2} + 2\pi > 0 \text{ so crit value is minimum}
\]

So max must occur at an end point

\[
\begin{align*}
  A(0) &= 100 \\
  A\left(\frac{20}{\pi}\right) &= \pi \left(\frac{20}{\pi}\right)^2 = \frac{400}{\pi}
\end{align*}
\]

\[
\text{Max: all wire goes to circle to max area}
\]
Find the shortest distance from the point \((3, 7)\) to the line \(y = 2x\).

Goal: Min \(d = \sqrt{(x-3)^2 + (y-7)^2}\)

Restriction: \(y = 2x\)

Goal Min \(d = \sqrt{(x-3)^2 + (2x-7)^2}\)

Look at \(d^2 = D = (x-3)^2 + (2x-7)^2\) critical value is at the same place!

\[D' = 2(x-3) + 2(2x-7)(2) = 0\]
\[2x-6 + 8x-28 = 0\]
\[10x = 34\]
\[x = \frac{17}{5}\]

Show min 2nd Deriv Test
\[D'' = 10 > 0\] so crit value is min

\[\text{shortest distance is } \sqrt{\left(\frac{17}{5}-3\right)^2 + \left(\frac{17}{5}-7\right)^2}\]

\[= \frac{1}{\sqrt{5}}\]
On Your Own:

Suppose $40\pi$ square feet of material are available to make a cylindrical can. Find the radius and height of the cylinder with the largest possible volume. (HINT: The lateral area of a cylinder is $A = 2\pi rh$. Don’t forget to include the top and bottom area as well!)

Goal: $\max V = \pi r^2 h$

Restriction: $S = 40\pi = 2\pi rh + 2\pi r^2$

$h = \frac{40\pi - 2\pi r^2}{2\pi r} = \frac{40 - 2r^2}{2r}$

Max $V = \pi r^2 \cdot \frac{40 - 2r^2}{2r} = \pi r(20 - r^2)$

$= 20\pi r - \pi r^3; \quad r \in [0, 5\sqrt{2}]$

$V' = 20\pi - 3\pi r^2 = 0$

$r^2 = \frac{20}{3}$

$r = \frac{2\sqrt{15}}{3}$ (Not in domain)

Show max 2nd Der Test
$V'' = -6\pi r < 0$ so critical is maximum

$r = \sqrt{\frac{20}{3}}$, $h = \frac{40 - 2\left(\frac{20}{3}\right)}{2\sqrt{\frac{20}{3}}}$ ft

$h$ simplified = $\frac{\sqrt{\frac{20}{3}}}{\frac{2\sqrt{20}}{3}} = \frac{2\sqrt{10}}{\sqrt{3}} = \frac{2\sqrt{30}}{3}$ ft