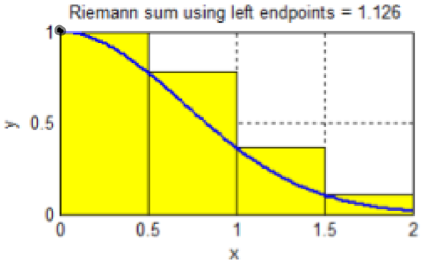
1 6.1-6.2: Area

Introduction:

The second historical problem of Calculus involves finding the area under the graph of a function. Suppose we want to find the area under the graph of $f(x) = e^{-x^2}$ between $x = 0$ and $x = 2$ (denoted by $\int_0^2 e^{-x^2}\,dx$. We cannot find the exact area geometrically (or even using techniques you may have previously learned in Calculus!), so we approximate the area using rectangles (see figures in class):

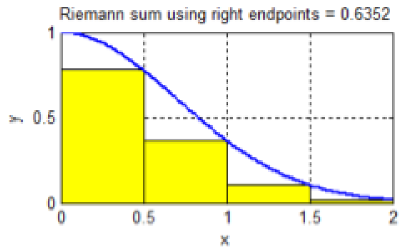


4 "left" rectangles

$A = 0.5f(0) + 0.5f(0.5) + 0.5f(1) + 0.5f(1.5)$

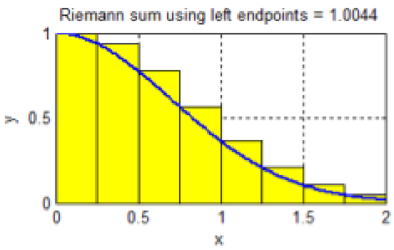
(6.1)

$A = \sum_{i=0}^{3} 0.5f(0.5i)$



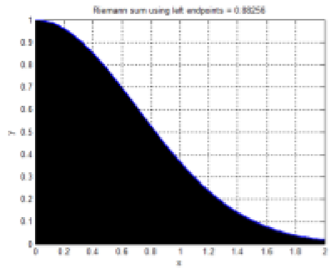
4 "right" rectangles

$A = \sum_{i=1}^{4} 0.5f(0.5i)$



8 "left" rectangles

$$A = \sum_{i=0}^{7} (0.25) f(0.25i)$$



512 left rectangles

What happens if more rectangles are used? more accurate results

How many rectangles are needed to get the exact area? ∞

Definitions:

partition: A partition P of an interval $[a,b]$ is a set of numbers $\{x_0, x_1, x_2, \ldots x_n\}$ with $a = x_0 < x_1 < x_2 < x_3 < \ldots < x_{n-1} < x_n = b$

Δx_i: $x_i - x_{i-1}$

$||P||$: (norm of the partition P) $=$ $\max(\Delta x_i)$

x_i^*: a given number such that $x_i^* \in [x_{i-1}, x_i]$

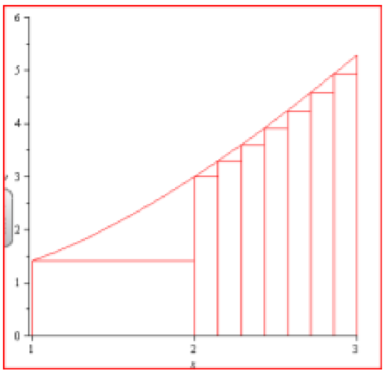
A Riemann Sum $= \displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x_i$

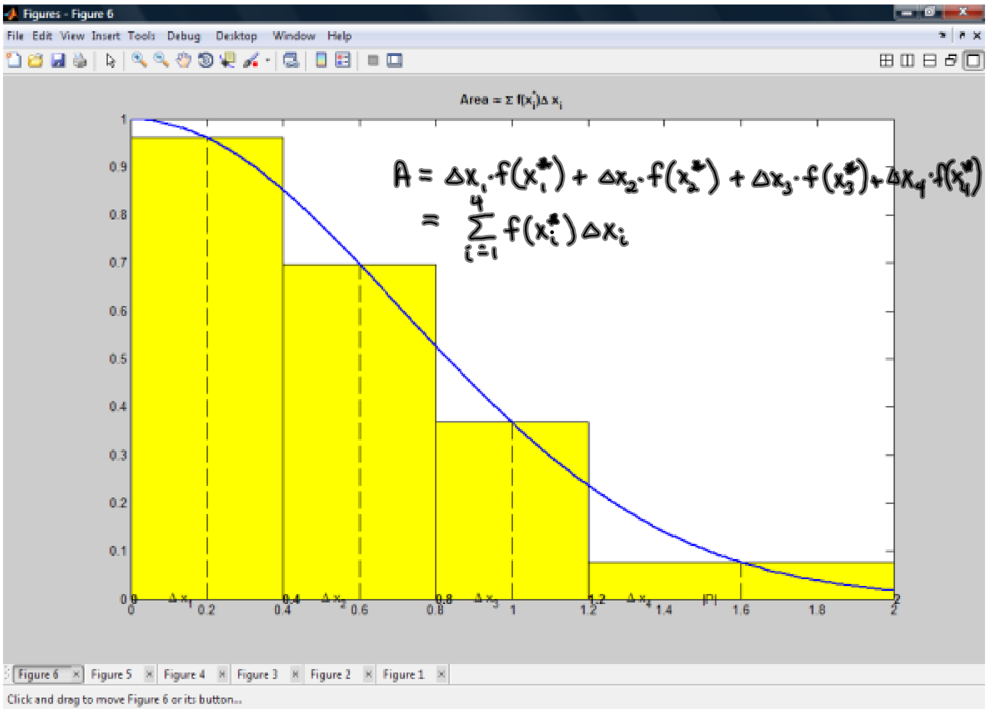
height · base

The **area under the graph** of a positive function f from $x = a$ to $x = b$:

$$\cancel{A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \,?}$$

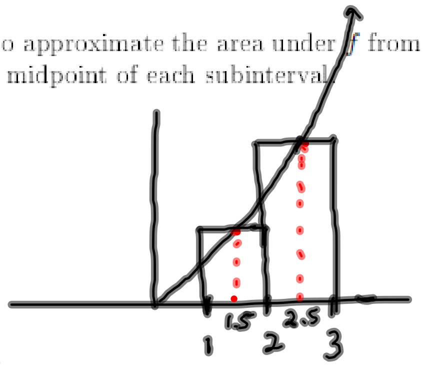
$$\boxed{\text{Area} = \lim_{|P|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i}$$




Figures - Figure 6
File Edit View Insert Tools Debug Desktop Window Help
Area = Σ f(x_i^*)Δ x_i
$A = \Delta x_1 \cdot f(x_1^*) + \Delta x_2 \cdot f(x_2^*) + \Delta x_3 \cdot f(x_3^*) + \Delta x_4 \cdot f(x_4^*)$
$= \sum_{i=1}^{4} f(x_i^*) \Delta x_i$
Figure 6 Figure 5 Figure 4 Figure 3 Figure 2 Figure 1
Click and drag to move Figure 6 or its button...

Examples:

Given $f(x) = x^2 + x$, write and compute a Riemann Sum to approximate the area under f from $x = 1$ to $x = 3$ using a partition $P = \{1, 2,$ [redacted] $, 3\}$. Let $x_i^* =$ the midpoint of each subinterval.



$$\text{Area} \approx \sum_{i=1}^{2} f(x_i^*)\Delta x_i$$

$$= \underbrace{f(1.5)\cdot 1}_{\text{height} \cdot \text{base}} + \underbrace{f(2.5)\cdot 1}_{\text{height} \cdot \text{base}}$$

$$= \boxed{(1.5^2 + 1.5)\cdot 1 + (2.5^2 + 2.5)\cdot 1}$$

$f(x) = x^2 + x$

Find the exact area under the graph of f from $x = 1$ to $x = 3$. (HINT: use n equally spaced ~~partitions~~ subintervals and take x_i^* =the right endpoint of each rectangle. Then let $n \to \infty$)

$$A = \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

$$= \lim_{|P| \to 0} \sum_{i=1}^{n} f(x_i^*) \frac{2}{n}$$

$$= \lim_{|P| \to 0} \sum_{i=1}^{n} \frac{2}{n} f\left(1 + \frac{2i}{n}\right)$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} f\left(1 + \frac{2i}{n}\right)$$

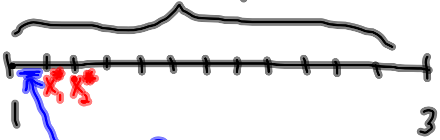
$$= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \left[\left(1 + \frac{2i}{n}\right)^2 + \left(1 + \frac{2i}{n}\right)\right]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2} + 1 + \frac{2i}{n}\right)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \left(2 + \frac{6i}{n} + \frac{4i^2}{n^2}\right)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{4}{n} + \frac{12i}{n^2} + \frac{8i^2}{n^3}\right)$$

n equal subintervals



$\Delta x_i = \frac{2}{n}$

$x_1^* = 1 + \frac{2}{n}$

$x_2^* = 1 + 2\left(\frac{2}{n}\right)$

$x_3^* = 1 + 3\left(\frac{2}{n}\right)$

$x_i^* = 1 + i\left(\frac{2}{n}\right)$

$|P| = \frac{2}{n} \to 0$ when $n \to \infty$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4}{n} + \frac{12i}{n^2} + \frac{8i^2}{n^3} \right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} + \sum_{i=1}^{n} \frac{12i}{n^2} + \sum_{i=1}^{n} \frac{8i^2}{n^3}$$

$$= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} 1 + \frac{12}{n^2} \sum_{i=1}^{n} i + \frac{8}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{4}{\cancel{n}} \cdot \cancel{n} + \frac{12}{n^{\cancel{2}}} \cdot \overset{\text{GIVEN}}{\frac{\cancel{n}(n+1)}{2}} + \frac{8}{n^{2}} \cdot \overset{\text{GIVEN}}{\frac{\cancel{n}(n+1)(2n+1)}{6}}$$

$$= 4 + 6 + \frac{16}{6}$$

$$= \boxed{\frac{38}{3}}$$