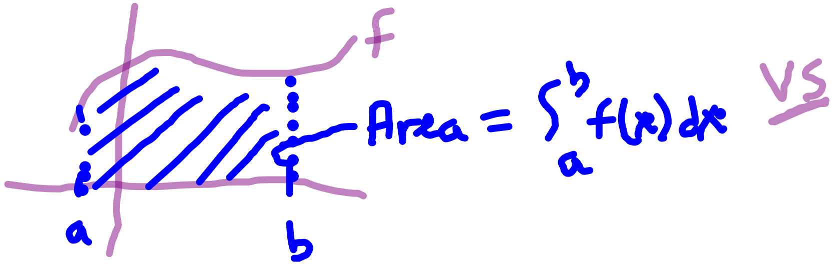
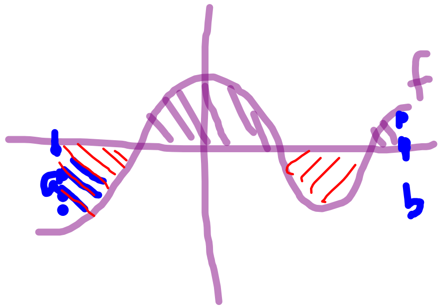
# 1 6.3: The Definite Integral

Note that, although we assumed $f$ was positive to illustrate the approximating rectangles, the definition can still be calculated even if $f$ is not always positive.

**The Definite Integral** of $f$ from $x = a$ to $x = b$ is given by $\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$

If $f > 0$, $\int_a^b f(x)\,dx$ gives us the area under the graph of $f$ from $x = a$ to $x = b$.




$\int_a^b f(x)dx$ = Area above − Area Below length $= b-a$

**Equally Spaced Partitions:** Let $n$ be the number of equally-spaced subintervals of $[a, b]$.

Then $\Delta x_i = \dfrac{b-a}{n}$

$a + i\,\Delta x$

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

and $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right)$

**Properties of Definite Integrals** (pp 383-385)

(NOTE: Some of the more useful properties for future sections are #2, 3, 5, and 8)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right)$$

$a = 0$
$b = 3$
$f(x) = x^2 - 3x + 1$

Given $f(x) = x^2 - 3x + 1$, find the exact value of $\int_0^3 f(x)\,dx$ from the definition.

$$\int_0^3 f(x)\,dx = \lim_{n\to\infty} \frac{3-0}{n} \sum_{i=1}^{n} f\left(0 + \frac{3-0}{n} i\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 3\left(\frac{3i}{n}\right) + 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{9i}{n} + 1\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \left(\sum_{i=1}^{n} \frac{9i^2}{n^2} - \sum_{i=1}^{n} \frac{9i}{n} + \sum_{i=1}^{n} 1\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2} - \frac{3}{n} \sum_{i=1}^{n} \frac{9i}{n} + \frac{3}{n} \sum_{i=1}^{n} 1$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum i^2 - \frac{27}{n^2} \sum i + \frac{3}{n} \sum 1$$

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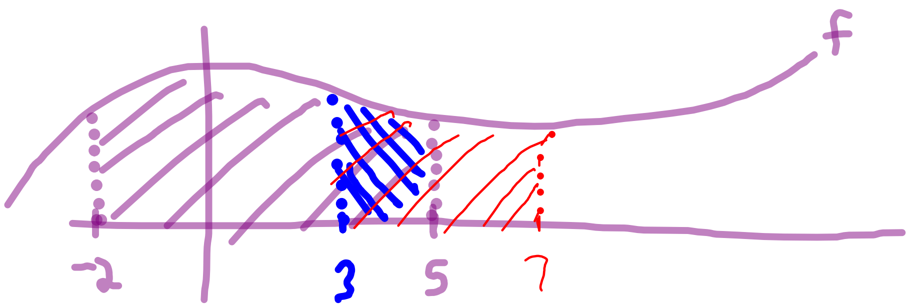
$$= \lim_{n\to\infty} \frac{27}{n^{3}} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{27}{n^2} \cdot \frac{n(n+1)}{2} + \frac{3}{n} \cdot n$$

$$= \lim_{n\to\infty} \frac{27(2n^2+3n+1)}{6n^2} - \frac{27(n+1)}{2n} + 3$$

$$= 9 - \frac{27}{2} + 3 = \boxed{-\frac{3}{2}}$$

Rewrite $\int_{-2}^{5} f(x)\,dx - \int_{3}^{5} f(x)\,dx + \int_{3}^{7} f(x)\,dx$ as a single integral.

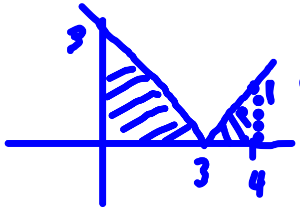



$= \int_{-2}^{7} f(x)\,dx$

**On Your Own**: Compute $\displaystyle\int_0^4 \left(|x-3| + \sqrt{16-x^2}\right)\,dx$

(HINT: Use properties to split up, then remember that integral = area since both functions are positive on $[0,4]$)

$$= \int_0^4 |x-3|\,dx + \int_0^4 \sqrt{16-x^2}\,dx$$




2 triangles

$$= \tfrac{1}{2}(3)(3) + \tfrac{1}{2}(1)(1) + \tfrac{1}{4}\cdot\pi(4)^2$$

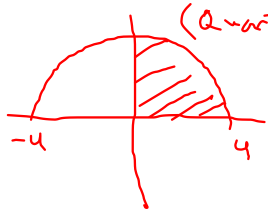
$$= \tfrac{9}{2} + \tfrac{1}{2} + 4\pi$$

$$= \boxed{5 + 4\pi}$$

$y^2 = \left(\sqrt{16-x^2}\right)^2$

$y^2 = 16 - x^2$

$x^2 + y^2 = 16$




(Quarter-circle)