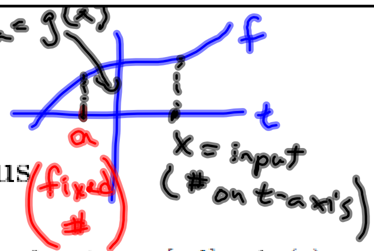
NOTE: $g(a)=0$



1 6.4-The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus, part I: Given f is a continuous function on $[a,b]$ and $g(x) = \int_a^x f(t)\,dt$, proof that g is differentiable:

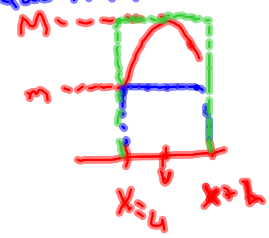
$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \underbrace{\int_x^{x+h} f(t)\,dt}_{\text{Look at}}$$ Properties of Integrals (6.3)

Claim

$m \le f(t) \le M$ on $[x, x+h]$ Extreme Value Thm



$$mh \le \int_x^{x+h} f(t)\,dt \le Mh$$ Property of Integrals (6.3)

$$m \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M$$

Let $f(u)=m$ and $f(v)=M$

$$f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)$$

As $h\to 0$ $u \to x$ so $f(u)\to f(x)$

$v \to x$ so $f(v) \to f(x)$

$\therefore \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$ by Squeeze Theorem

So $g'(x) = f(x)$

Fundamental Theorem of Calculus, part II: If F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$

NOTE:

$$\int_1^3 (x^2+x)\,dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_1^3$$

$$= \left(\frac{1}{3}\cdot 27 + \frac{1}{2}\cdot 9\right) - \left(\frac{1}{3}\cdot 1 + \frac{1}{2}\cdot 1\right)$$

$$= 9 + \frac{9}{2} - \frac{1}{3} - \frac{1}{2}$$

$$= \frac{38}{3}$$

Pf

Let $g(x) = \int_a^x f(t)\,dt$

g is an antiderivative of f (so is F)

$\therefore \quad g(x) = F(x) + C$ since $g(a) = 0$

$0 = F(a) + C$

$C = -F(a)$

$$\int_a^b f(x)\,dx = g(b) = \boxed{F(b) - F(a)}$$

Examples:

Find $F'(x)$ given:

$$F(x) = \int_{\pi}^{x} \frac{\sin t}{t}\, dt$$

Part I of FTC (derivative functions $\leftrightarrow$ integral functions)

inverse operations

$$\boxed{F'(x) = \frac{\sin x}{x}}$$

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$$F(x) = \int_{x^2}^{3} \sqrt{1+t^3}\, dt$$

Method I (Part I of FTC) Chain Rule

$$F(x) = -\int_3^{x^2} \sqrt{1+t^3}\, dt$$

$$= -\sqrt{1+(x^2)^3} \cdot 2x$$

$$= \boxed{-2x\sqrt{1+x^6}}$$

Method II (use Part II of FTC)

Let $g(t)$ be an antiderivate of $\sqrt{1+t^3}$.

$$F(x) = g(3) - g(x^2)$$

$$F'(x) = 0 - g'(x^2) \cdot 2x$$

$$= -\sqrt{1+(x^2)^3} \cdot 2x$$

$$= \boxed{-2x\sqrt{1+x^6}}$$

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Compute $\int_1^2 \left(x+\frac{1}{x^2}\right)^2 dx$ $\cancel{2x^{-1+1}}$ x^{-4}

$$= \int_1^2 \left(x^2 + \frac{2}{x} + \frac{1}{x^4}\right) dx$$

$$= \frac{1}{3}x^3 + 2\ln|x| + \frac{1}{-3}x^{-3} \Big|_1^2$$

$$= \left(\frac{8}{3} + 2\ln 2 - \frac{1}{3}\cdot 2^{-3}\right) - \left(\frac{1}{3} + 2\ln 1 - \frac{1}{3}\cdot 1^{-3}\right)$$

$$= \boxed{\frac{8}{3} + 2\ln 2 - \frac{1}{3}\cdot 2^{-3} - \left(\frac{1}{3} - \frac{1}{3}\cdot 1^{-3}\right)}$$

$$= \frac{8}{3} + 2\ln 2 - \frac{1}{24} = \frac{63}{24} + 2\ln 2 = \boxed{\frac{21}{8} + 2\ln 2}$$

Compute $\int_0^\pi (\sin x + x^3 - e^x)\,dx$

$$= -\cos x + \frac{1}{4}x^4 - e^x \Big|_0^\pi$$

$$= \left(-\cos\pi + \frac{1}{4}\pi^4 - e^\pi\right) - \left(-\cos 0 + \frac{1}{4}\cdot 0^4 - e^0\right)$$

$$= \boxed{\left(1 + \frac{\pi^4}{4} - e^\pi\right) - (-1-1)}$$

$$= \boxed{3 + \frac{\pi^4}{4} - e^\pi} \text{ OR}$$

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On Your Own: Find the exact area under the graph of $f(x) = \dfrac{2}{1+x^2}$ from $x = 0$ to $x = \sqrt{3}$.

$2 \cdot \frac{1}{1+x^2}$ derivative of $\arctan x$

$$\int_0^{\sqrt{3}} \frac{2}{1+x^2}\,dx$$

$$2 \arctan x \Big|_0^{\sqrt{3}}$$

$$= 2 \arctan \sqrt{3} - 2 \arctan 0$$

$$= \boxed{\frac{2\pi}{3}}$$

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