3.11-Linear Approximation and Differentials

Purpose: To understand differentials and linear approximations to a function near a certain point.

Definitions: Given \( y = f(x) \), the differential \( dy \) represents an independent quantity (a small change in \( x \)). Then the differential \( dy \) is given by:

\[ \Delta y = f'(a) \Delta x \]

Seemingly Unrelated Topic: Recall graphing \( y = \sin x \). What happened as you zoom in on the point corresponding to \( x = 0 \)? The graph looks like its tangent line at \( x = 0 \) \( (y = x) \)

Idea: The tangent line approximates the curve \( y = f(x) \) near \( x = a \).

What is the equation of the line tangent to \( y = f(x) \) at the point where \( x = a \)?

**Step:** \( f'(a) \) at \( x = a \), \( m = f'(a) \)

**Point:** \( (a, f(a)) \)

**Equation:** \( y = f(a) + f'(a)(x-a) \)

Or: \( y = f(a) + f'(a)(x-a) \)
Definition: The Linear Approximation (or Linearization) of \( f \) at \( x = a \) is

\[
L(x) = f(a) + f'(a)(x-a)
\]

The Connection:

Goal: approximate \( f(x) \) near \( x = a \)

\[
\Delta y = f(x) - f(a) \approx L(x) - f(a)
\]

\[
\Delta y \approx L(x) - f(a)
\]

\[
\Delta y \approx f(a) + f'(a)(x-a) - f(a)
\]

\[
\Delta y = f'(a)(x-a)
\]

\[
\Delta y = f'(a)dx = dy
\]

Examples:

Given \( y = \sqrt{x} \), find \( \Delta y \) and \( dy \) if \( x = 4 \) and \( \Delta x = dx = 1 \).

\[
dy = f'(a)dx
\]

\[
dy = \frac{1}{2}(4)^{-\frac{1}{2}}(1)
\]

\[
= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}
\]

\[
f'(x) = \frac{1}{2} x^{-\frac{1}{2}}
\]

\[
\Delta y = \sqrt{5} - \sqrt{4}
\]

\[
\approx .2361
\]

So \( \sqrt{5} \approx 2.25 \)

\[
2 + \frac{1}{4} \approx f(a) + \Delta y
\]
Use differentials to approximate \( \cos 62^\circ \)

\[
dy = f'(a) \, dx
\]

\[
a = \frac{\pi}{3}
\]

\[
f'(a) = -\sin \left( \frac{\pi}{3} \right)
\]

\[
f(x) = \cos x
\]

\[
f'(x) = -\sin x
\]

\[
dy = \frac{-\sqrt{3}}{2} \cdot \frac{\pi}{90} = \frac{-\sqrt{3} \pi}{180}
\]

\[
dx = \frac{26}{180} = \frac{26}{180}
\]

So \( \cos 62^\circ \approx \cos \frac{26 \pi}{180} - \frac{\pi}{3} \cdot \frac{26}{180} \)

Check: \( \frac{1}{2} - \frac{\sqrt{3} \pi}{180} \approx 0.4698 \)

\( \cos (62^\circ) \approx 0.4695 \)
Find the linear approximation of \( f(x) = \sqrt{x} \) at \( a = \frac{9}{4} \) and use it to approximate \( \sqrt{2} \).

\[
L(x) = f(a) + f'(a)(x-a) \quad a = \frac{9}{4} \\
L(x) = \sqrt{\frac{9}{4}} + \frac{1}{2}(\frac{9}{4})^{\frac{1}{2}}(x-\frac{9}{4}) \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \\
L(x) = \frac{3}{2} + \frac{3}{4}(x-\frac{9}{4}) \quad \text{Equation of the line to } f(x) = \sqrt{2} \text{ at } a = \frac{9}{4} \\
L(2) = \frac{3}{2} + \frac{3}{4}(2-\frac{9}{4}) \\
= \frac{3}{2} + \frac{3}{4}(-\frac{1}{4}) = \frac{3}{2} - \frac{3}{12} = \frac{17}{12}
\]
The circumference around the middle of a sphere is measured to be 40 cm, with a possible error of ±1 cm. Use differentials to estimate the possible error in the volume of the sphere.

\[ C = 40, \quad \therefore \quad r = \frac{40}{2\pi} = \frac{20}{\pi} \]

\[ V = \frac{4}{3} \pi r^3 \]

\[ dV = 4\pi r^2 \, dr \]

\[ = 4\pi \left( \frac{300}{\pi^2} \right) \left( \pm \frac{1}{2\pi} \right) \]

\[ = \pm \frac{3000}{\pi^2} \approx 39 \text{ cm}^2 \]

\[ V(C) : \quad V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi \left( \frac{C}{2\pi} \right)^3 \]

\[ V = \frac{1}{6\pi^2} C^3 \]

\[ dV = \frac{2}{2\pi^2} C^2 \, dC \]

\[ \text{NOTE : relative error} \quad \frac{dV}{V} = \frac{\frac{3000}{\pi^2} (\pm 1)}{\left( \frac{40}{\pi} \right)^3} = \frac{\pm 800}{\pi^2} \]

\[ \frac{dV}{V} = \frac{800}{\pi^2 \cdot \left( \frac{40}{\pi} \right)^3} = 0.075 \text{ or } 7.5\% \]
A better approximation:

\[ Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \]

Why more accurate?

**Linear**
- \( L(a) = f(a) \)
- \( L'(a) = f'(a) \)
- But \( L''(a) = 0 \)

**Quadratic**
- \( Q(a) = f(a) \)
- \( Q'(a) = f'(a) \)
- \( Q''(a) = f''(a) \)