5.3-Derivatives and the Shapes of Curves

Mean Value Theorem: \( \text{if } f \text{ is continuous on } [a, b] \text{ and differentiable on } (a, b), \text{ then there is a } c \in (a, b) \text{ such that } f'(c) = \frac{f(b) - f(a)}{b - a} \text{ slope} \)

Recall 5.1: What \( f' \) and \( f'' \) say about \( f \):
- \( f' > 0 \rightarrow f \text{ inc} \)
- \( f' < 0 \rightarrow f \text{ dec} \)
- \( f'' > 0 \rightarrow f' \text{ inc } \rightarrow f \text{ conc up} \)
- \( f'' < 0 \rightarrow f' \text{ dec } \rightarrow f \text{ conc down} \)

If \( f \) changes concavity at \( x = c \), then \( x = c \) is an inflection point of \( f \) \((f''(c) = 0 \text{ if it exists})\)

Second Derivative Test: If \( f \) has a critical value at \( x = c \), and:
- \( f''(c) > 0 \), then \( f \) has a relative minimum at \( x = c \)
- \( f''(c) < 0 \), then \( f \) has a relative maximum at \( x = c \)
**Examples:**

Determine where the function \( f(x) = x^3 - 3x^2 + 1 \) is increasing and decreasing, concave up, and concave down.

**inc/dec:** Find critical values:

\[
f'(x) = 3x^2 - 6x = 0\]

\[
3x(x-2) = 0
\]

\[
x = 0 \quad x = 2
\]

\[
\text{Test } x = 1 \quad \text{Test } x = 3
\]

\[
f \text{ inc } (-\infty, 0) \cup (2, \infty)
\]

\[
f \text{ dec } (0, 2)
\]

**NOTE:** At no extra charge, we know:

rel max at \( x = 0 \) \((0, 1)\)

rel min at \( x = 2 \) \((2, -3)\)

**conc:** Find critical values of \( f' \) \((f'' = 0 \text{ or DNE})\)

\[
f''(x) = 6x - 6 = 0
\]

\[
f'' = \begin{cases} - & \text{Test } x = 0 \\ + & \text{Test } x = 2 \end{cases}
\]

\[
f \text{ conc up } (1, \infty)
\]

\[
f \text{ conc down } (-\infty, 1)
\]

**NOTE:** At no extra charge, we know:

inf pt at \( x = 1 \) \((1, -1)\)
Determine where the function \( f(x) = x^2 e^{-2x} \) is increasing, decreasing, concave up, and concave down.

\[
f'(x) = x^2 (-2e^{-2x}) + 2xe^{-2x} = 0
\]

\[
2xe^{-2x} (-x + 1) = 0
\]

\[
x = 0 \quad \text{or} \quad x = 1
\]

\( f' \) is increasing in \((0,1)\), decreasing in \((-\infty,0) \cup (1,\infty)\).

Relative max at \( x = 1 \) \((1,1e^{-2})\)

Relative min at \( x = 0 \) \((0,0)\)

\[
f''(x) = (-2x^2 + 2x) e^{-2x}
\]

\[
(-4x^2 + 4x + 2) e^{-2x} = 0
\]

\[
2e^{-2x} (4x^2 - 4x - 4x + 2) = 0
\]

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{4 - 4(4)(-4)}}{2(2)} = \frac{4 \pm \sqrt{32}}{4} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}
\]

\( f'' \) is positive in \((-\infty, \frac{2-\sqrt{2}}{2}) \cup (\frac{2+\sqrt{2}}{2}, \infty)\)

\( f'' \) is negative in \( (\frac{2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}) \)

\( f \) is concave up in \((-\infty, \frac{2-\sqrt{2}}{2}) \cup (\frac{2+\sqrt{2}}{2}, \infty)\)

\( f \) is concave down in \( (\frac{2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}) \)
Find the inflection points of \( f(x) = -x^2 \cos x + 6 \cos x + 4x \sin x, \quad x \in [-\pi, \pi]. \)

\[
\begin{align*}
f'(x) &= -x^2 (-\sin x) + \cos x (-2x) - 6 \sin x + 4 \cos x + 4 \sin x \\
      &= x^2 \sin x + 2x \cos x - 2 \sin x \\
n''(x) &= x^2 \cos x + 2x \sin x + 2x(-\sin x) + 2 \cos x - 2 \cos x \\
n'''(x) &= x^2 \cos x = 0 \\
\end{align*}
\]

\( x=0, \quad x=\frac{\pi}{2}, \quad \frac{3\pi}{2} \)

\[
\begin{array}{c|c|c|c}
& - & + & - \\
\hline
x & 0 & \frac{\pi}{2} & \frac{3\pi}{2} \\
\hline
f' & & & \\
\hline
f'' & & & \\
\hline
\end{array}
\]

Inflection points: \( x = \frac{-\pi}{2} \) and \( x = \frac{3\pi}{2} \)